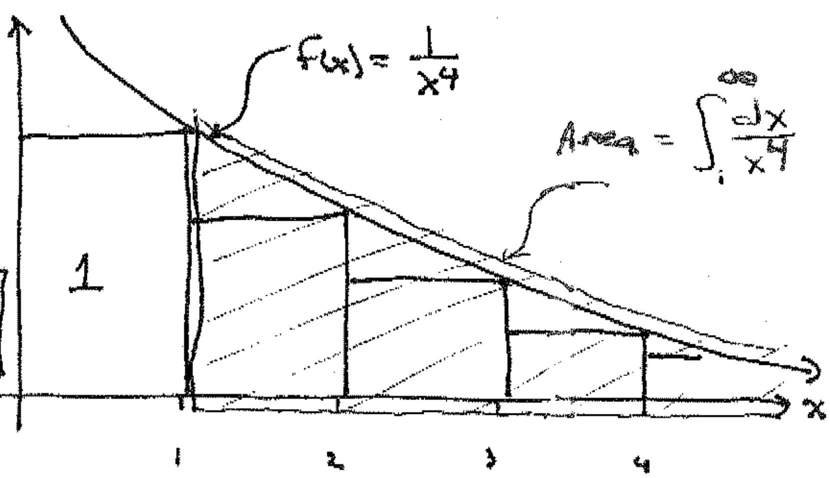
11.3: Integral Test & Estimating Sums

Ex1: $\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

Reminds us of the convergent $\int_1^{\infty} \frac{dx}{x^4} = \lim_{t\to\infty} -\frac{1}{3x^3}\Big|_1^t \quad \frac{1}{3}$

Since I believe it will converge, I use right endpoints.

The series is below the function

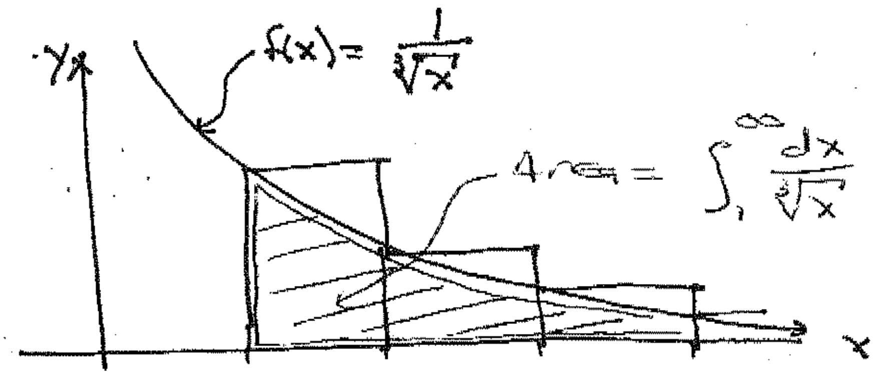


AND we have $0 < \sum_{n=1}^{\infty} \frac{1}{n^4} < 1 + \frac{1}{3}$. Hence it converges. Using Mathematica, we know that it converges to $\pi^4/90$. But, most importantly it converges.

Ex2: $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \dots$

This reminds us of the divergent $\int_1^{\infty} \frac{dx}{\sqrt[3]{x}}$.

Since I believe the series will diverge, I use left endpoints.



The ~~series~~ sequence is above the function

and $\int_1^\infty \frac{dx}{\sqrt[3]{x}} < \sum_{n=1}^\infty \frac{1}{\sqrt[3]{n}}$. So the series diverges.

<u>The Integral Test</u> Suppose f is a cont, pos, decreasing fct on $1 \le x$ and let $a_n = f(n)$. Then $\sum_{n=1}^\infty a_n$ is convergent iff $\int_1^\infty f(x)\,dx$ is convergent.

<u>Ex 3:</u> Does $\sum_{n=1}^\infty n e^{-n}$ converge?

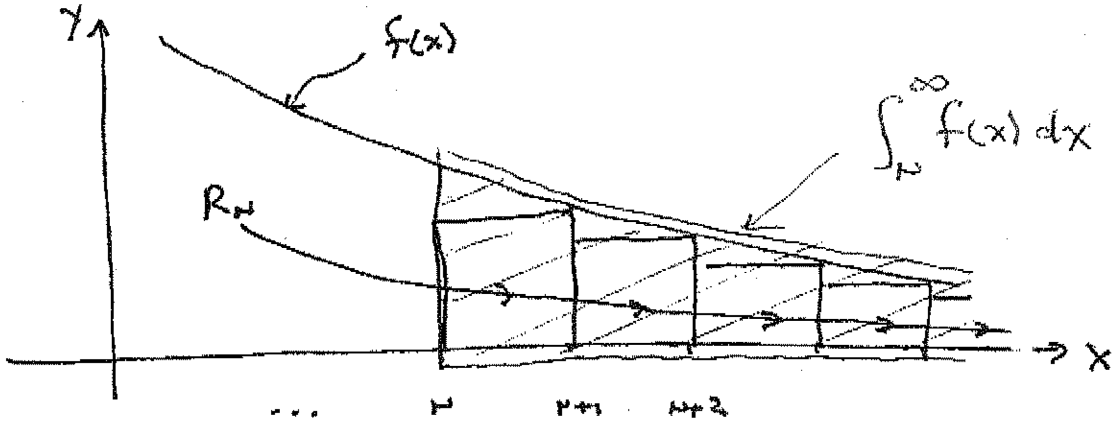
<u>Ex 4:</u> For what values of p does $\sum_{n=1}^\infty \frac{1}{n^p}$ converge?

<u>Estimating Sums</u>

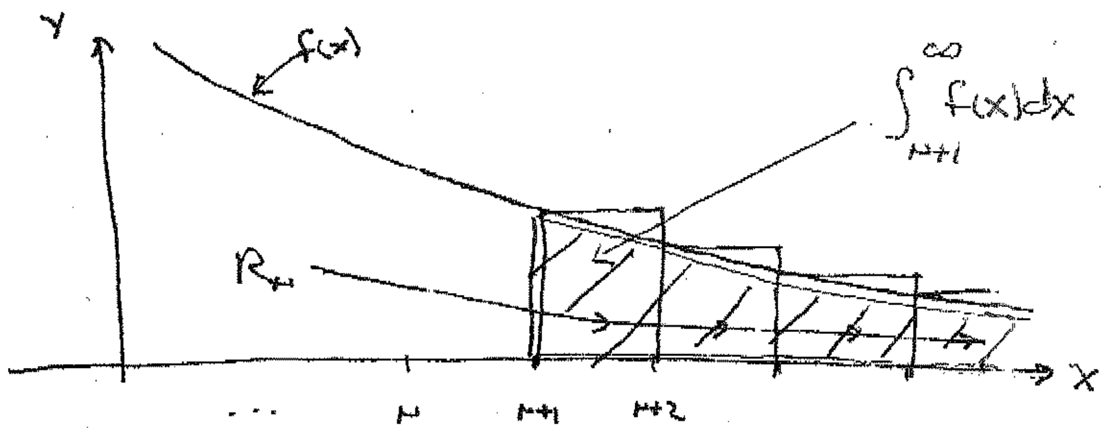
Suppose we know $\sum a_n$ converges but we cannot evaluate it directly. So, we want to estimate the series w/ the n^{th} partial sum s_n. But, this leaves us w/ an important question.

What is the error associated w/ each partial sum?

Remainder: $R_n = a_{n+1} + a_{n+2} + \ldots$ in S_n.



and $R_n \leq \int_n^\infty f(x)\,dx$. similarly



and $\int_{n+1}^\infty f(x)\,dx \leq R_n$.

<u>Remainder Estimate for the Integral Test</u>

If $\sum a_n$ converges by the I.T. and $R_n = S - S_n$, then $\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx$.

Ex 5: Suppose $\sum_{n=1}^{\infty} \frac{1}{n^5} = S.$

a) find S_5

b) bound R_5

c) For what n's is $R_n \leq 0.00005$?

a) $\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \approx 1.036662.$

b) $\underbrace{\int_6^{\infty} \frac{dx}{x^5}} \leq R_n \leq \int_5^{\infty} \frac{dx}{x^5}$

$\Rightarrow \lim_{t \to \infty} \left[-\frac{1}{4x^4}\right]_6^t \leq R_n \leq \lim_{t \to \infty} \left[-\frac{1}{4x^4}\right]_5^t$

$\Rightarrow \frac{1}{4 \cdot 6^4} \leq R_n \leq \frac{1}{4 \cdot 5^4}$

c) $R_n \leq \underbrace{\int_n^{\infty} \frac{dx}{x^5}} \leq 0.00005$

AND $\underbrace{\lim_{t \to \infty} \left[-\frac{1}{4x^4}\right]_n^t}$

$\Rightarrow R_n \leq \frac{1}{4n^4} \leq 0.00005$

so, solve $\frac{1}{4n^4} \leq 0.00005$

$\Rightarrow \frac{1}{4(0.00005)} \leq n^4$

$\Rightarrow 8.41 \leq n$

So, so long as $n \geq 9$, R_n will be less than 0.00005.