10 Review

Concept Check

1. (a) What is a parametric curve?
   (b) How do you sketch a parametric curve?
2. (a) How do you find the slope of a tangent to a parametric curve?
   (b) How do you find the area under a parametric curve?
3. Write an expression for each of the following:
   (a) The length of a parametric curve
   (b) The area of the surface obtained by rotating a parametric curve about the x-axis
4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.
   (b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.
   (c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?
5. (a) How do you find the slope of a tangent line to a polar curve?
   (b) How do you find the area of a region bounded by a polar curve?
   (c) How do you find the length of a polar curve?
6. (a) Give a geometric definition of a parabola.
   (b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?
7. (a) Give a definition of an ellipse in terms of foci.
   (b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
8. (a) Give a definition of a hyperbola in terms of foci.
   (b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
   (c) Write equations for the asymptotes of the hyperbola in part (b).
9. (a) What is the eccentricity of a conic section?
   (b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?
   (c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

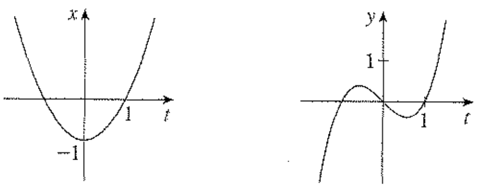
1. If the parametric curve $x = f(t)$, $y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent when $t = 1$.
2. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then
$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$$
3. The length of the curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, is $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$.
4. If a point is represented by (x, y) in Cartesian coordinates (where $x \neq 0$) and (r, θ) in polar coordinates, then $\theta = \tan^{-1}(y/x)$.
5. The polar curves $r = 1 - \sin 2\theta$ and $r = \sin 2\theta - 1$ have the same graph.
6. The equations $r = 2$, $x^2 + y^2 = 4$, and $x = 2 \sin 3t$, $y = 2 \cos 3t$ $(0 \leqslant t \leqslant 2\pi)$ all have the same graph.
7. The parametric equations $x = t^2$, $y = t^4$ have the same graph as $x = t^3$, $y = t^6$.
8. The graph of $y^2 = 2y + 3x$ is a parabola.
9. A tangent line to a parabola intersects the parabola only once.
10. A hyperbola never intersects its directrix.

Exercises

1–4 Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

1. $x = t^2 + 4t, \quad y = 2 - t, \quad -4 \leqslant t \leqslant 1$
2. $x = 1 + e^{2t}, \quad y = e^t$
3. $x = \cos\theta, \quad y = \sec\theta, \quad 0 \leqslant \theta < \pi/2$
4. $x = 2\cos\theta, \quad y = 1 + \sin\theta$

5. Write three different sets of parametric equations for the curve $y = \sqrt{x}$.
6. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



7. (a) Plot the point with polar coordinates $(4, 2\pi/3)$. Then find its Cartesian coordinates.
   (b) The Cartesian coordinates of a point are $(-3, 3)$. Find two sets of polar coordinates for the point.
8. Sketch the region consisting of points whose polar coordinates satisfy $1 \leqslant r < 2$ and $\pi/6 \leqslant \theta \leqslant 5\pi/6$.

9–16 Sketch the polar curve.

9. $r = 1 - \cos\theta$
10. $r = \sin 4\theta$
11. $r = \cos 3\theta$
12. $r = 3 + \cos 3\theta$
13. $r = 1 + \cos 2\theta$
14. $r = 2\cos(\theta/2)$
15. $r = \dfrac{3}{1 + 2\sin\theta}$
16. $r = \dfrac{3}{2 - 2\cos\theta}$

17–18 Find a polar equation for the curve represented by the given Cartesian equation.

17. $x + y = 2$
18. $x^2 + y^2 = 2$

19. The curve with polar equation $r = (\sin\theta)/\theta$ is called a **cochleoid**. Use a graph of r as a function of θ in Cartesian coordinates to sketch the cochleoid by hand. Then graph it with a machine to check your sketch.
20. Graph the ellipse $r = 2/(4 - 3\cos\theta)$ and its directrix. Also graph the ellipse obtained by rotation about the origin through an angle $2\pi/3$.

21–24 Find the slope of the tangent line to the given curve at the point corresponding to the specified value of the parameter.

21. $x = \ln t, \; y = 1 + t^2; \quad t = 1$
22. $x = t^3 + 6t + 1, \quad y = 2t - t^2; \quad t = -1$
23. $r = e^{-\theta}; \quad \theta = \pi$
24. $r = 3 + \cos 3\theta; \quad \theta = \pi/2$

25–26 Find dy/dx and d^2y/dx^2.

25. $x = t + \sin t, \quad y = t - \cos t$
26. $x = 1 + t^2, \quad y = t - t^3$

27. Use a graph to estimate the coordinates of the lowest point on the curve $x = t^3 - 3t$, $y = t^2 + t + 1$. Then use calculus to find the exact coordinates.
28. Find the area enclosed by the loop of the curve in Exercise 27.
29. At what points does the curve

$$x = 2a\cos t - a\cos 2t \qquad y = 2a\sin t - a\sin 2t$$

have vertical or horizontal tangents? Use this information to help sketch the curve.

30. Find the area enclosed by the curve in Exercise 29.
31. Find the area enclosed by the curve $r^2 = 9\cos 5\theta$.
32. Find the area enclosed by the inner loop of the curve $r = 1 - 3\sin\theta$.
33. Find the points of intersection of the curves $r = 2$ and $r = 4\cos\theta$.
34. Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.
35. Find the area of the region that lies inside both of the circles $r = 2\sin\theta$ and $r = \sin\theta + \cos\theta$.
36. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

37–40 Find the length of the curve.

37. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 2$
38. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$
39. $r = 1/\theta, \quad \pi \leqslant \theta \leqslant 2\pi$
40. $r = \sin^3(\theta/3), \quad 0 \leqslant \theta \leqslant \pi$

Graphing calculator or computer required CAS Computer algebra system required

41–42 Find the area of the surface obtained by rotating the given curve about the x-axis.

41. $x = 4\sqrt{t}, \quad y = \dfrac{t^3}{3} + \dfrac{1}{2t^2}, \quad 1 \leqslant t \leqslant 4$

42. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

43. The curves defined by the parametric equations

$$x = \frac{t^2 - c}{t^2 + 1} \qquad y = \frac{t(t^2 - c)}{t^2 + 1}$$

are called **strophoids** (from a Greek word meaning "to turn or twist"). Investigate how these curves vary as c varies.

44. A family of curves has polar equations $r^a = |\sin 2\theta|$ where a is a positive number. Investigate how the curves change as a changes.

45–48 Find the foci and vertices and sketch the graph.

45. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

46. $4x^2 - y^2 = 16$

47. $6y^2 + x - 36y + 55 = 0$

48. $25x^2 + 4y^2 + 50x - 16y = 59$

49. Find an equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

50. Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

51. Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$.

52. Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

53. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and that has its other focus at the origin.

54. Show that if m is any real number, then there are exactly two lines of slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and their equations are $y = mx \pm \sqrt{a^2m^2 + b^2}$.

55. Find a polar equation for the ellipse with focus at the origin, eccentricity $\frac{1}{3}$, and directrix with equation $r = 4 \sec \theta$.

56. Show that the angles between the polar axis and the asymptotes of the hyperbola $r = ed/(1 - e \cos \theta)$, $e > 1$, are given by $\cos^{-1}(\pm 1/e)$.

57. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$$

(a) Show that if (a, b) lies on the curve, then so does (b, a); that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?
(b) Find the points on the curve where the tangent lines are horizontal or vertical.
(c) Show that the line $y = -x - 1$ is a slant asymptote.
(d) Sketch the curve.
(e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.
(f) Show that the polar equation can be written in the form

$$r = \frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}$$

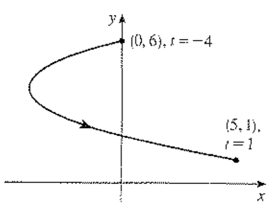
(g) Find the area enclosed by the loop of this curve.
CAS (h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

True-False Quiz

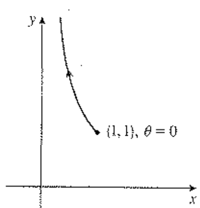
1. False 3. False 5. True 7. False 9. True

Exercises

1. $x = y^2 - 8y + 12$

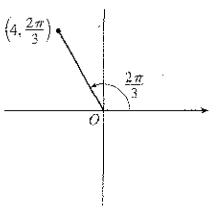


3. $y = 1/x$



5. $x = t,\ y = \sqrt{t};\ x = t^4,\ y = t^2;$
$x = \tan^2 t,\ y = \tan t,\ 0 \leqslant t < \pi/2$

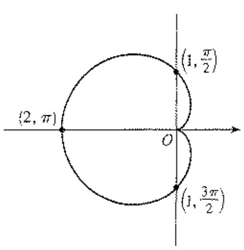
7. (a)



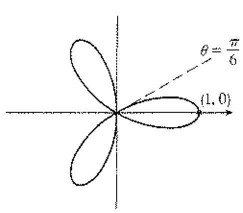
$(-2, 2\sqrt{3})$

(b) $(3\sqrt{2}, 3\pi/4)$, $(-3\sqrt{2}, 7\pi/4)$

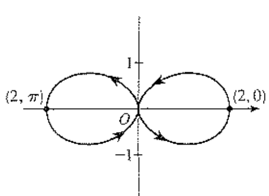
9.



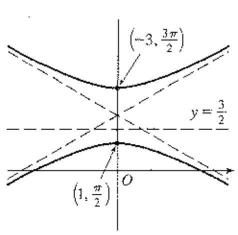
11.



13.

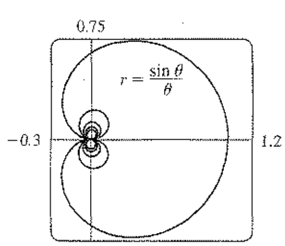


15.



17. $r = \dfrac{2}{\cos\theta + \sin\theta}$

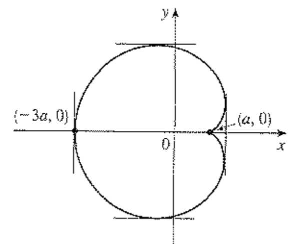
19.



21. 2 23. -1

25. $\dfrac{1 + \sin t}{1 + \cos t}, \dfrac{1 + \cos t + \sin t}{(1 + \cos t)^3}$ 27. $\left(\frac{11}{8}, \frac{3}{4}\right)$

29. Vertical tangent at $\left(\frac{3}{2}a, \pm\frac{1}{2}\sqrt{3}\,a\right), (-3a, 0)$; horizontal tangent at $(a, 0), \left(-\frac{1}{2}a, \pm\frac{3}{2}\sqrt{3}\,a\right)$



31. 18 33. $(2, \pm\pi/3)$ 35. $\frac{1}{2}(\pi - 1)$

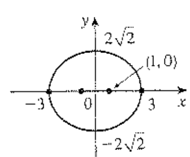
37. $2(5\sqrt{5} - 1)$

39. $\dfrac{2\sqrt{\pi^2 + 1} - \sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right)$

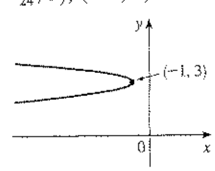
41. $471{,}295\pi/1024$

43. All curves have the vertical asymptote $x = 1$. For $c < -1$, the curve bulges to the right. At $c = -1$, the curve is the line $x = 1$. For $-1 < c < 0$, it bulges to the left. At $c = 0$ there is a cusp at $(0, 0)$. For $c > 0$, there is a loop.

45. $(\pm 1, 0), (\pm 3, 0)$



47. $\left(-\frac{25}{24}, 3\right), (-1, 3)$



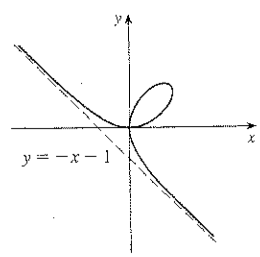
49. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ 51. $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$

53. $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$ 55. $r = \dfrac{4}{3 + \cos\theta}$

57. (a) At $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$

(b) Horizontal tangents at $(0, 0)$ and $(\sqrt[3]{2}, \sqrt[3]{4})$; vertical tangents at $(0, 0)$ and $(\sqrt[3]{4}, \sqrt[3]{2})$

(d)



(g) $\frac{3}{2}$