13.3: Arclength and Curvature

Recall from 10.2 that arclength $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

The arclength of the space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

is $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$

$\quad = \int_a^b |\vec{r}'(t)|\, dt$

this is assuming the curve is traversed exactly once

on $a \leq t \leq b$.

Ex1: Find the length of $\vec{r}(t) = \langle 3\sin(t), 7t, -3\cos(t) \rangle$

on $-5 \leq t \leq 5$.

If c is a piecewise smooth curve given by

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on $a \leq t \leq b$ and at least

one of the functions x, y, z is $1-1$ on (a,b),

then the arclength function s is:

$$s(t) = \int_a^t |\vec{r}'(u)|\, du$$

Differentiating w/respect to t gives

$$\frac{ds}{dt} = s'(t) = |\vec{r}'(t)|$$

EX2: Reparametrize $\vec{r}(t) = \langle 3\sin(t), 7t, -3\cos(t)\rangle$ writ the arclength measured from $(0,0,-3)$ in the direction of increasing t.

soln:

$(0,0,-3) \Longleftrightarrow t=0$.

recall from (ex1) that $|\vec{r}'(t)| = \sqrt{58} = \dfrac{ds}{dt}$

$\Rightarrow \quad s = \displaystyle\int_0^t \sqrt{58}\, du$

$\qquad = \sqrt{58}\,t$
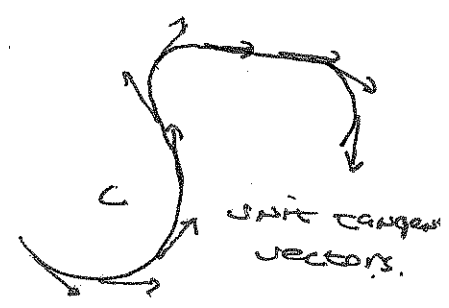
$\Rightarrow \quad t = \dfrac{s}{\sqrt{58}}$

so, $r[t(s)] = \left\langle 3\sin\left(\dfrac{s}{\sqrt{58}}\right), \dfrac{7s}{\sqrt{58}}, -3\cos\left(\dfrac{s}{\sqrt{58}}\right)\right\rangle$ //.

Recall that the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

Defn: The curvature k of a curve is

$$k = \left|\dfrac{d\vec{T}}{ds}\right| \qquad (D.f_n \#1)$$

This should make intuitive sense.

unit tangent vectors.

This 1se. defn of k is extremely difficult. So notice we can't use the chain rule

$$\dfrac{d\vec{T}}{ds} = \dfrac{d\vec{T}}{dt}\dfrac{dt}{ds} = \dfrac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \dfrac{\vec{T}'(t)}{|r'(t)|}$$

and $\quad k = \dfrac{|\vec{T}'(t)|}{|r'(t)|} \qquad (D.f_n \#2)$

our goal is to find an expression for the osculating circle ... so we need the relationship between the curvature & radius of a circle.

Ex 3: Show that the curvature of a circle w/radius a is $k = \frac{1}{a}$.

We want an even simpler way to calculate/work w/curvature... but we need 4 facts.

#1 $\quad \vec{v} \times \vec{v} = \vec{0}$

#2 $\quad \frac{d}{dt}\left[f(t) \, \vec{u}(t) \right] = f'(t) \, \vec{u}(t) + f(t) \, \vec{u}'(t)$

$$\text{(product rule)}$$

#3 $\quad$ If $|\vec{r}(t)| = c$ (a constant), then $\vec{r}'(t) \perp \vec{r}(t)$ for all t.

$\square$ proof.

$$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$$

$\Rightarrow \vec{r}(t) \cdot \vec{r}(t) = c^2$

differentiate both sides using the product rule

$\Rightarrow \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$

$\Rightarrow 2\vec{r}'(t) \cdot \vec{r}(t) = 0$

$\Rightarrow \vec{r}'(t) \cdot \vec{r}(t) = 0$

Hence $\vec{r}(t)$ & $\vec{r}'(t)$ are always orthogonal.

#4 $\quad \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

Now we are ready to derive our formula

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} \Rightarrow \vec{r}' = |\vec{r}'|\,\vec{T}$$

$$\Rightarrow \vec{r}' = \frac{ds}{dt}\,\vec{T} \quad (Fact \#_2)$$

Differentiate both sides using the product rule on R.H.S.

$$\Rightarrow \vec{r}'' = \frac{d^2s}{dt^2}\,\vec{T} + \frac{ds}{dt}\,\vec{T}'$$

$$\Rightarrow \vec{r}' \times \vec{r}'' = \frac{ds}{dt}\,\vec{T} \times \left(\frac{d^2s}{dt^2}\,\vec{T} + \frac{ds}{dt}\,\vec{T}'\right) \quad (Fact \#4)$$

$$\overset{(Fact \#1)}{=} \frac{ds}{dt}\frac{d^2s}{dt^2}(\vec{T} \times \vec{T}) + \left(\frac{ds}{dt}\right)^2(\vec{T} \times \vec{T}')$$

$$= \left(\frac{ds}{dt}\right)^2(\vec{T} \times \vec{T}')$$

$$\Rightarrow |\vec{r}' \times \vec{r}''| = \left(\frac{ds}{dt}\right)^2 |\vec{T} \times \vec{T}'| \quad \begin{array}{l}(Fact \#3) \\ since\ \vec{T} \perp \vec{T}'\end{array}$$

$$= \left(\frac{ds}{dt}\right)^2 |\vec{T}||\vec{T}'|$$

$$= \left(\frac{ds}{dt}\right)^2 |\vec{T}'|$$

Thus $|\vec{T}'| = \dfrac{|\vec{r}' \times \vec{r}''|}{\left(\frac{ds}{dt}\right)^2}$

$$= \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^2}$$

recall $k = \dfrac{|\vec{T}'|}{|\vec{r}'|}$

> If you have a plane curve $y = f(x)$
> $$k(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$ (Dfn #4)

so $k = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$ (Dfn #3)

Ex 4: Consider $\vec{r}(t) = \langle \cos t, \sin t, \ln(\cos t)\rangle$

Find the curvature.

Normal vector: since $|\vec{T}(t)| = 1$, $\vec{T} \cdot \vec{T}' = 0$

So we define the unit normal vector

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

and we define the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Ex4 rev.: (a) Find $\vec{T}, \vec{N}, \vec{B}$

(b) Find $\vec{T}, \vec{N}, \vec{B}$ @ $(1,0,0)$

(c) eqt of the osculating plane.

(d) eqt of the kissing circle...

$\vec{r}(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle$ $t = 0 + 2k\pi, k \in \mathbb{Z}$

$\vec{r}'(t) = \langle -\sin t, \cos t, -\tan t \rangle$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$

$= \sqrt{1 + \tan^2 t}$

$= |\sec t|$

$\vec{r}''(t) = \langle -\cos t, -\sin t, -\sec^2 t \rangle$

Find k after evaluating $\vec{r}'(0)$ and $\vec{r}''(0)$.

$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -s & c & -t \\ -c & -s & -\sec^2 t \end{vmatrix}$

$= \langle -\sec t - \frac{\sin^2 t}{\cos t}, -\frac{\sin t}{\cos^2 t} + \sin t, 1 \rangle$

$= \langle -\frac{1 + \sin^2 t}{\cos t}, \sin t(1 - \sec^2 t), 1 \rangle \Big|_{t=0} \quad \langle -1, 0, 1 \rangle$

$|\vec{r}'(t) \times \vec{r}''(t)| \Big|_{t=0} = \sqrt{2}$

$$\hookrightarrow \quad k = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \bigg|_{t=0} \quad \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\vec{T}(t) = \frac{1}{|\sec t|} \langle -\sin t, \cos t, -\tan t \rangle$$

$$= \langle -\tan t, 1, \frac{-\sin t}{\cos^2 t} \rangle \bigg|_{t=0} \quad \langle 0, 1, 0 \rangle$$

$$\vec{T}'(t) = \langle -\sec^2 t, 0, \frac{-\cos t \cdot \cos^2 t - 2\cos t \cdot (-\sin t)(-\sin t)}{\cos^4 t} \rangle$$

$$= \langle -\sec^2 t, 0, -\sec t - \frac{2\sin^2 t}{\cos^3 t} \rangle \bigg|_{t=0} \quad \langle -1, 0, -1 \rangle$$

$$|\vec{T}'(0)| = \sqrt{2}$$

$$\vec{N}(t) = \langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle$$

$$\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$$

osculating plane: $-\frac{1}{\sqrt{2}}(x-1) + 0(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$

kissing circle w/Radius $R = \frac{1}{k}$

$$\frac{1}{k}\vec{T}\cos\theta + \frac{1}{k}\vec{N}\sin\theta + \vec{r}(t) + \frac{1}{k}\vec{N}(t)$$

$$\Rightarrow \vec{r}(\theta) = \frac{1}{\sqrt{2}}\langle 0, 1, 0 \rangle\cos\theta + \frac{1}{\sqrt{2}}\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle\sin\theta$$

$$+ \langle 1, 0, 0 \rangle + \frac{1}{\sqrt{2}}\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \rangle$$