11.1: Sequences

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A is an ordered list $a_1, a_2, \ldots, a_n, \ldots$

common notations include

i) $\{a_1, a_2, a_3, \ldots\}$

ii) $\{a_n\}$

iii) $\{a_i\}_{i=1}^{\infty}$

w/ corresponding examples

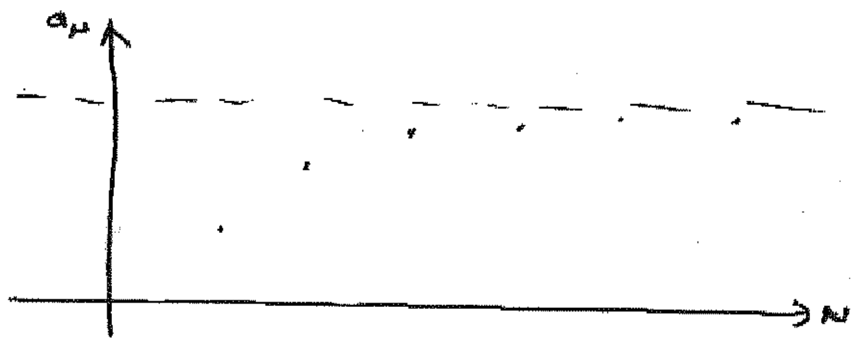
i) $\{1, 1, 2, 3, 5, \ldots\}$

ii) $\left\{\frac{(-1)^n}{n}\right\}$

iii) $\left\{\frac{n!}{n^n}\right\}_{n=2}^{\infty}$

Ex 1: Find a_n if $\{a_n\} = \left\{\frac{-2}{4}, \frac{3}{8}, \frac{-4}{16} \cdots\right\}$

What is the behavior of $a_n = \frac{n}{n+1}$ as $n \to \infty$. Written another way, $\lim_{n\to\infty} \frac{n}{n+1}$.

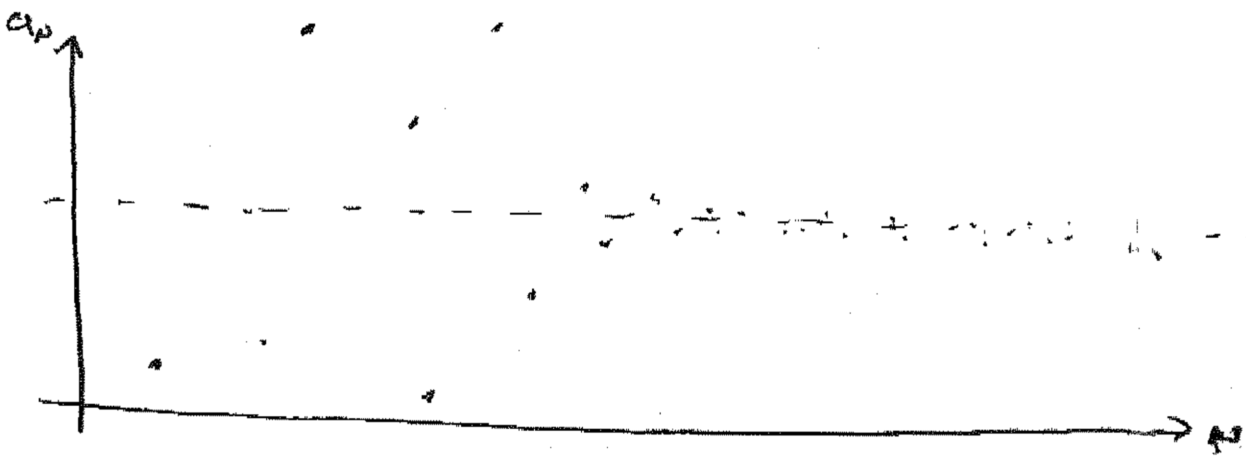


what is the domain

Definition: A sequence $\{a_n\}$ has the limit L if we can make the terms a_n arbitrarily close to L for sufficiently large n. Then we write $\lim_{n\to\infty} a_n = L$. Else, a_n diverges.

more precisely,

<u>Definition</u>. A sequence $\{a_n\}$ converges to the limit L if $\forall \epsilon > 0 \; \exists N$ s.t. $|a_n - L| < \epsilon$ when $n > N$.



The only difference between this defn. and our old defn of the limit of a fct is the domain.

<u>Theorem</u>: If $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$ when $n \in \mathbb{Z}$, then $\lim_{n \to \infty} a_n = L$.

Begin here day 2

<u>Definition</u>. $\lim_{n \to \infty} a_n = \infty \Rightarrow \forall M > 0 \; \exists N \in \mathbb{Z}$ s.t. $n > N \Rightarrow a_n > M$.

recall $\forall$
$\exists$

clarification ϵ epsilon
$\in$ "is an element of."

The <u>limit laws</u>, if $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

ii) $\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n$

iii) $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, $\lim_{n\to\infty} b_n \neq 0$.

v) $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

The Squeeze Theorem for sequences

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

<u>theorem</u>: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

<u>Ex2</u>: $\lim_{n\to\infty} \frac{\sin(n)}{n}$

l'Hopital's Rule on p 308.

Ex 3: Does $\lim_{n\to\infty} \sin(n)$ converge or diverge?

Ex 4: Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n^2}$ if it exists.

Ex 5: Evaluate $\lim_{n\to\infty} \frac{n!}{n^n}$ if it exists.

$$\frac{n!}{n^n} = \frac{1}{n}\cdot\underbrace{\left(\frac{2\cdot 3\cdot \ldots n}{n\cdot n\cdot \quad n}\right)}_{<1}$$

so $0 < \frac{n!}{n^n} < \frac{1}{n}$ AND the limit converges by the squeeze theorem.

ONLY IF time (ha).
Ex 6: Find the radius of convergence for $\{r^n\}$

Defn. $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for $n \geq 1$ and decreasing if $a_n > a_{n+1}$. $\{a_n\}$ is monotonic if it is increasing or decreasing.

Ex 7: show $\{\ln(n)\}$ is increasing.
(use the derivative).
this can also be done using log rules.

<u>Defn.</u> $\{a_n\}$ is bounded above if $\exists M$ s.t. $a_n \leq M \ \forall n \geq 1$.

bounded below. $a_n \geq m$.

If $\{a_n\}$ is bounded above and below, then we say it is a bounded sequence.

NOTE: upperbound vs. least upper bound.

<u>Thm:</u> Every bounded, monotonic sequence is convergent.

see proof in book ... completeness axiom.

NOTE: see CD for hints on red problems.

Hint: #65 & #72 are similar.