Test 3
Dusty Wilson
Math 220

No work = no credit
No Calculators

Name: Key

Thus, the task is, not so much to see what no one has yet seen; but to think what nobody has yet thought, about that which everybody sees.

Erwin Rudolf Josef Alexander Schrödinger
1887 – 1961 (Austrian physicist)

Warm-ups (1 pt each)[1]: $\vec{e}_1^{\,T}\vec{e}_1 =$ 1 $\quad -1^2 =$ -1 $\quad \vec{e}_1\vec{e}_1^{\,T} =$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

1.) (1 pt) According to Schrödinger, what is necessary to complete the task? (See above). Answer using complete English sentences.

The key is to have wherewithall to contemplate what is familiar & yet unknown.

2.) (10 pt) Find the eigenvalues of $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$ and determine the stability of the zero state of the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.

solve $\det(A - \lambda I) = 0$

$$\Rightarrow 0 = (5-\lambda)(4-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= (5-\lambda)(4-\lambda)\Big[\underbrace{(3-\lambda)(1-\lambda)}_{3-4\lambda+\lambda^2} + 2\Big]$$

$$3 - 4\lambda + \lambda^2 + 2$$

$$= (5-\lambda)(4-\lambda)(\lambda^2 - 4\lambda + 5)$$

$$\lambda = 4, 5, \quad \lambda = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= 2 \pm i$$

$|\lambda| > 1$ so the system is unstable

[1] In the warm-ups, $\vec{e}_i$ refers to the standard basis vector in $\mathbb{R}^2$.

14

24

Find the QR factorization of $A = \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}$

① $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\|\vec{v}_1\| = 3$ and $\vec{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

5 pts

② $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix} - 9 \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

$\|\vec{v}_2^{\perp}\| = \sqrt{4 + 16 + 16} = 6$ and $\vec{u}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

7 pts

③ $\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$

$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 14/9 \\ -14/9 \\ 7/9 \end{bmatrix}$

$\|\vec{v}_3^{\perp}\| = \sqrt{\frac{196}{81} + \frac{196}{81} + \frac{49}{81}} = \sqrt{\frac{441}{81}} = \frac{21}{9} = \frac{7}{3}$ and $\vec{u}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

$$A = \overset{Q}{\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}} \overset{R}{\begin{bmatrix} 3 & 9 & 1/3 \\ 0 & 6 & 2/3 \\ 0 & 0 & 7/3 \end{bmatrix}}$$

Test 3
Dusty Wilson
Math 220

No work = no credit

Name: Key

Thus, the task is, not so much to see what no one has yet seen; but to think what nobody has yet thought, about that which everybody sees.

Erwin Rudolf Josef Alexander Schrödinger
1887 – 1961 (Austrian physicist)

the distribution vector

1.) (10 pts) The matrix $A=\begin{bmatrix} .5 & .25 \\ .5 & .75 \end{bmatrix}$ has eigenvalues 1 and 0.25. If $\vec{x}_0=\begin{bmatrix} .7 \\ .3 \end{bmatrix}$, find $\lim_{t\to\infty}\left(A^t\vec{x}_0\right)$

$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \Rightarrow \vec{x}_{equ} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$\Rightarrow \lim_{t\to\infty} A^t \vec{x}_0 = \vec{x}_{equ} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

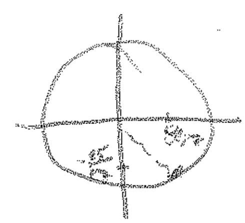
2.) (10 pts) Consider the rotation-scaling matrix $B=\begin{bmatrix} 8 & 15 \\ -15 & 8 \end{bmatrix}$. Find the angle of rotation (degrees or radians) and the scaling factor.

to 2 decimals:

$$B = 17\begin{bmatrix} 8/17 & 15/17 \\ -15/17 & 8/17 \end{bmatrix}$$

$\cos\theta = 8/17$

$\sin\theta = -15/17$



$$\Rightarrow \tan\theta = -\frac{15}{8}$$

$$\Rightarrow \theta = \tan^{-1}\left(-\frac{15}{8}\right)$$

$= -61.93^\circ$ or -1.08 rad

3.) (10 pts) Consider $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

a.) Find both eigenvectors for given that $\lambda_{1,2} = 2 \pm i$.

$A - (2+i)I = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \quad \frac{1}{1-i}R_1 \to R_1$

$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2}+\frac{1}{2}i \\ -2 & -1-i \end{bmatrix} \quad 2R_1 + R_2 \to R_2$

$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2}+\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$

eigenvector

$\begin{bmatrix} -\frac{1}{2}-\frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + i\begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$

so this means the two vecs are

$\overset{p}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}} \pm i \overset{q}{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}$

b.) Find an invertible matrix S such that $S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a and b are real numbers.

$S = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

4.) (10 pts) What are the similarities and differences between an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$ and an eigenbasis $\vec{v}_1, \dots, \vec{v}_m$ for a subspace V?

similarities

① span ✓

② L I

differences

① $\vec{u}_1, \dots, \vec{u}_m$ are orthogonal w/ length 1 while $\vec{v}_1, \dots, \vec{v}_n$ most likely are not.

② $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1, \dots, m$ while $A\vec{u}_i \neq \lambda \vec{u}_i$ for most A & $\vec{v}_i$'s

5.) (9 pts) Answer the following. 1 point per problem is for an explanation/justification:

a.) If A_{nxn} is diagonalizable (over $\mathbb{R}$), then there must be an eigenbasis for $\mathbb{R}^n$.

True since A is diagonalizable iff the # of L.I. is n (Thm 7.3.3).

b.) There exists a real 5x5 matrix without any real eigenvalues.

False, eigen vecs come in conjugate pairs

c.) If α is an eigenvalue of A_{nxn} and β is an eigenvalue of B_{nxn}, then $\alpha\beta$ is an eigenvalue of AB.

False, unless A, B have the same eigenvector.

6.) (10 pts) Find a real closed formula for the trajectory of $\vec{x}(t+1) = A^t\vec{x}(t)$, where $A = \begin{bmatrix} 1 & 5 \\ -2 & 7 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

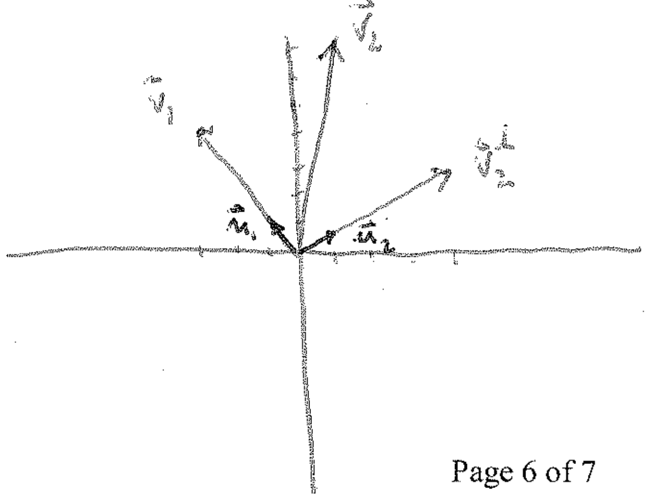
$A = S B S^{-1}$ w/ $S = \begin{bmatrix} 1/2 & 3/2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & +1 \\ -1 & 4 \end{bmatrix}$

CUT.

7.) (10 pts) Perform Gram-Schmidt orthogonalization on $\vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and then illustrate your work with a sketch.

$\hat{u}_1 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$

$\vec{v}_2^{\perp} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} - 5\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\hat{u}_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$



8.) (10 pts) Prove that for every vector $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$ we can write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ where $\vec{x}^{\parallel}$ is in V and $\vec{x}^{\perp}$ is perpendicular to V.

proof.

Let $\vec{x} \in \mathbb{R}^n$ an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$ for V be given.

If $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, then there exist constants $c_1, \dots, c_m$

s.t. $x^{\parallel} = c_1 \vec{u}_1 + \dots + c_m \vec{u}_m$

$\Rightarrow \vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$

$= \vec{x} - c_1 \vec{u}_1 - \dots - c_m \vec{u}_m$ is orthogonal to $\vec{u}_1, \dots, \vec{u}_m$

$\Rightarrow \vec{u}_i \cdot \vec{x}^{\perp} = 0$ for $i = 1, \dots, m$

$\Rightarrow 0 = \vec{u}_i \cdot (\vec{x} - c_1 \vec{u}_1 - \dots - c_m \vec{u}_m)$

$\Rightarrow c_i = \vec{u}_i \cdot \vec{x}$ for $i = 1, \dots, m$

since these constants exist, we have a unique formula for $\vec{x}^{\parallel}$, $\vec{x}^{\perp}$, and $\vec{x}$.

Q.E.D.