$\bar{x} = 68\%$

med = 67.2%

high = 106.6%

Test 2b
Dusty Wilson
Math 220

No work = no credit

Name: key

Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.

Evariste Galois
1811 – 1832 (French mathematician)

Warm-ups (1 pt each)[1]: $\vec{e}_2 \cdot \vec{e}_1 =$ 0 $\quad \vec{e}_1^{\,T}\vec{e}_1 = \begin{bmatrix} 1 \end{bmatrix}$ $\quad \vec{e}_2\,\vec{e}_2^{\,T} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

1.) (1 pt) According to Galois, how do authors most hurt their readers? Answer using complete English sentences.

They conceal/hide the tough parts.

2.) (10 pts) The matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ has eigenvalues $\lambda = -2$ and $\lambda = 4$.

Alg mult 2

Find the corresponding (a.) ~~eigenvectors~~, (b.) eigenspaces, and (c.) the geometric multiplicity of each eigenvalue.

$\lambda = -2$: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. $\quad E_{\lambda=-2} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

geometric mult = 2

$\lambda = 4$: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. $\quad E_{\lambda=4} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$

geo. mult = 1

[1] In the warm-ups, $\vec{e}_i$ refers to the standard basis vector in $\mathbb{R}^2$. Vectors with a T are written as row vectors, not column vectors

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 3 & 4 & 7 & 8 \end{bmatrix}$$

3.) (10 pts) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 9 & 6 \\ 1 & 2 & 4 & 1 \end{bmatrix}$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a.) Find a basis for the image of A.

basis: $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

$x_1 = -x_3$

$x_2 = -x_3 - 2x_4$

$x_3 = x_3$

$x_4 = x_4$

b.) Find the kernel of A.

$$\ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

c.) rank(A) = 2 and nullity(A) = 2

4.) (4 pts) Answer the following:

a.) True or False, if vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$, then they must form a basis of $\mathbb{R}^n$.

True.

b.) True or False, there exists a 5x4 matrix whose image consist of all of $\mathbb{R}^5$.

$A_{5\times 4} \vec{x}_{4\times 1}$

False.

rank + nullity = 4. $\neq 5$

5.) (~~10~~ 7 pts) A subset W of the vector space $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it has the following properties. Explain the meaning of any terms/phrases used when defining these properties.

(1) $\vec{0} \in W$

(2) if $\vec{x}, \vec{y} \in W$ then $\vec{x}+\vec{y} \in W$
closed under addition.

(3) if $\vec{x} \in W$ and $k \in \mathbb{R}$ then $k\vec{x} \in W$
closed under scalar mult.

6.) (10 pts) For the matrix $A_{n\times n}$, there are at least 9 statements equivalent to, "A is invertible." List at least five of them. List more for 1 extra credit point (each).

i.) A is invertible.

ii.) $A\vec{x} = \vec{b}$ has unique soln $\vec{x}$ for all $\vec{b} \in \mathbb{R}^n$.

iii.) $rref(A) = I$

iv.) $rank(A) = n$

v.) $im(A) = \mathbb{R}^n$

vi.) $ker(A) = \{\vec{0}\}$

vii.) (1 pt extra credit) $det(A) \neq 0$

viii.) (1 pt extra credit) col. vectors of A form a basis of $\mathbb{R}^n$

ix.) (1 pt extra credit) col vectors of A span $\mathbb{R}^n$.

x.) (1 pt extra credit) col vecs are L.I.

7.) (10 pts) Consider the linear transformation T such $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = -\vec{v}_1 - \vec{v}_2$ where $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$S = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$$

a.) Find the matrix $\boldsymbol{B}$ of the linear transformation

$$B = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

b.) If $\vec{x} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$, find $[\vec{x}]_B$

$$[\vec{x}]_B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

c.) For the given $\vec{x}$, find $[T(\vec{x})]_B$

$$[T(\vec{x})]_B = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

d.) For the given $\vec{x}$, find $T(\vec{x})$

$$S B S^{-1} \vec{x} = T(\vec{x}) = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

8.) (10 pts) We say $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ are a <u>basis</u> for a subspace V of $\mathbb{R}^n$ if they have the following properties. Explain the meaning of any terms/phrases used when defining these properties.

They span V: for all $\vec{v} \in V$, there exists constants $c_1, \ldots, c_m$ s.t. $c_1\vec{v}_1 + \ldots + c_m\vec{v}_n = \vec{v}$.

They are linearly independent: $c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{0}$ has only the trivial soln.