7.6: Stability

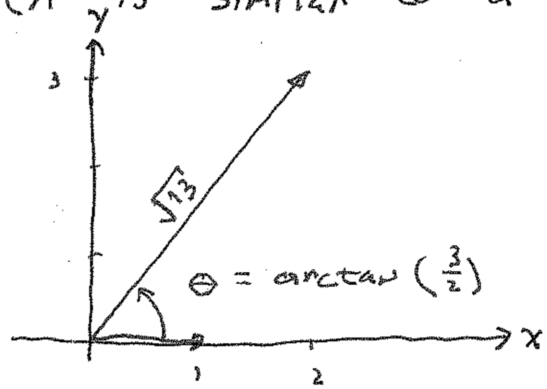
What happens to Dynamical Systems as $t \to \infty$.

ex1: recall $A = \begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix}$

$$\lambda = 2 \pm 3i$$

$$S^{-1} A S = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad \text{where} \quad S = \begin{bmatrix} -1 & -3 \\ 0 & 5 \end{bmatrix}$$

(A is similar to a rotation - scaling matrix).



rotate by θ

scale by $\sqrt{13}$

so $A = S \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} S^{-1}$

$$= \sqrt{13}\, S \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} S^{-1}$$

and $A^t = \sqrt{13}^{\,t}\, S \begin{bmatrix} \cos(t\theta) & -\sin(t\theta) \\ \sin(t\theta) & \cos(t\theta) \end{bmatrix} S^{-1}$

If $\vec{X}_0 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, find a real closed formula for $\vec{X}(t+1) = A\vec{x}(t)$.

much easier to find the closed form RSL

That is, we want $\vec{X}(t) = A^t x_0$

$$\overbrace{A^t \vec{X}_0}$$

t	x	y
0	-2	5
1	8	-5
2	58	-85
3	128	-275
4	-242	5
5	-2632	3595
6	-7382	14315

$$= \sqrt{13}^t S \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \underbrace{S^{-1}\vec{X}_0}_{[X_0]_B}$$

(see note)

$$= \sqrt{13}^t S \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \sqrt{13}^t \begin{bmatrix} -1 & -3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -c-s \\ -s+c \end{bmatrix}$$

$$= \sqrt{13}^t \begin{bmatrix} c+s+3s-3c \\ -5s+5c \end{bmatrix}$$

$$= \sqrt{13}^t \begin{bmatrix} 4s - 2c \\ 5c - 5s \end{bmatrix}$$

$$= \underbrace{\sqrt{13}^t \begin{bmatrix} 4\sin(t\theta) - 2\cos(t\theta) \\ 5\cos(t\theta) - 5\sin(t\theta) \end{bmatrix}}$$

a real closed formula for $\vec{X}(t)$. Clearly this is not a stable system.

The scaling factor makes it tough to find a window

Calculator notes

- $\arctan(3/2) \mapsto \theta$
- parametric mode
- check your graph by discarding $(\sqrt{13}^t)$ to start w/. You should get an ellipse.
- Add $(\sqrt{13})^t$ in and check against the table.

Note: $S^{-1}\vec{X}_0$ is $\vec{X}_0$ in terms of the new coordinate system. From $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ we have that $\vec{X}_0 = -1\vec{v}_1 + 1\vec{v}_2$ and so $S^{-1}\vec{X}_0 = [\vec{X}_0]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thm: Consider a dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.
where A is a real 2×2 matrix w/eigenvalues
$\lambda_{1,2} = p \pm iq$, $q \neq 0$. Let $r = |\lambda_1| = |\lambda_2| = \sqrt{p^2+q^2}$.

$r=1$: $\vec{x}(t)$ lies on an ellipse.

$r>1$ $\vec{x}(t)$ spirals out.

$r<1$ $x(t)$ spirals in to the origin.

ex 2:

We quote from a text on computer graphics (M. Beeler et al., "HAKMEM," MIT Artificial Intelligence Report AIM-239, 1972):

> Here is an elegant way to draw almost circles on a point-plotting display.
>
> ```
> CIRCLE ALGORITHM:
>
> NEW X = OLD X - K*OLD Y;
> NEW Y = OLD Y + K*NEW X.
> ```
>
> This makes a very round ellipse centered at the origin with its size determined by the initial point. The circle algorithm was invented by mistake when I tried to save a register in a display hack!

(In the preceding formula, k is a small number.) Here, a dynamical system is defined in "computer lingo." In our terminology, the formulas are

$$\begin{aligned} x(t+1) &= x(t) - ky(t), \\ y(t+1) &= y(t) + kx(t+1). \end{aligned}$$

a. Find the matrix of this transformation. (Note the entry $x(t+1)$ in the second formula.)

b. Explain why the trajectories are ellipses, as claimed.

(a) $A = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix}$ check

(b) NTS: A has complex eigenvalues w/ magnitude one.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -k \\ k & (1-\lambda)-k^2 \end{vmatrix}$$

$$= \lambda^2 + \lambda(k^2-2) + 1$$

$$\Rightarrow \quad \lambda = \frac{(2-k^2) \pm \sqrt{(k^2-2)^2 - 4}}{2}$$

$$= \frac{2-k^2 \pm \sqrt{k^2(k^2-4)}}{2}$$

order changes when i is factored out.

λ is complex if $|k| < 2$.

$$\Rightarrow \quad \lambda = p \pm qi \quad \text{where } p = \frac{2-k^2}{2} \text{ and } q = \sqrt{\frac{k^2(4-k^2)}{2}}$$

$$\Rightarrow \quad |\lambda| = \sqrt{p^2+q^2} = 1$$

$\therefore$ $\vec{x}(t)$ lives on an elliptic trajectory when $|k| < 2$.

Key concepts from linear alg.

(1) Find or approx. solutions to linear systems.

(2) Linear transformation

- subspaces
- change of basis to simplify the linear trans.
- dynamical systems.

We are searching in the light.

Additional example.

$$A = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$$

eigenvals.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-3-\lambda) + 10$$

$$= \lambda^2 - 9 + 10$$

$$= \lambda^2 + 10$$

$$\Rightarrow \lambda = \pm i$$

eigenvecs

$$\begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix} \quad \frac{1}{3-i} R_1 \to R_1 \qquad \frac{-5}{3-i} \cdot \frac{3+i}{3+i} = \frac{-15-5i}{10} = -\frac{3}{2} - \frac{1}{2}i$$

$$\begin{bmatrix} 1 & -\frac{3}{2} - \frac{1}{2}i \\ 2 & -3-i \end{bmatrix} \quad R_2 - 2R_1 \to R_2$$

$$\begin{bmatrix} 1 & -\frac{3}{2} - \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$$ so the eigenvec is $\begin{bmatrix} \frac{3}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$

and both are $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

so $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = S^{-1} \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix} S$ where $S = \begin{bmatrix} 1/2 & 3/2 \\ 0 & 1 \end{bmatrix}$

what rotation-scaling matrix is this?

factor: $\sqrt{0^2 + 1^2} = 1.$

rotation factor : $\frac{\pi}{2}$

Find the angle & scaling factor

scaling factor: $r = \sqrt{a^2+b^2} = \sqrt{17}$

angle : $\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{1}{4}\right)$

NOTE: check the quadrant.

$$\text{so} \quad \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} = \sqrt{17}\begin{bmatrix} \cos\left(\overbrace{\tan^{-1}\left(\frac{1}{4}\right)}^{\theta}\right) & -\sin\left(\tan^{-1}\left(\frac{1}{4}\right)\right) \\ \sin\left(\tan^{-1}\left(\frac{1}{4}\right)\right) & \cos\left(\tan^{-1}\left(\frac{1}{4}\right)\right) \end{bmatrix}$$

$$= \sqrt{17}\begin{bmatrix} \frac{4}{\sqrt{17}} & -\frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}^t = \sqrt{17}^t \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix}$$

Find $A^t \vec{x}_0$ by calculating R → L

$$\text{so} \quad \vec{x}(t) = A^t \vec{x}_0 = S \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}^t S^{-1} \vec{x}_0$$

$$= \sqrt{17}^t \, S \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} \left[\vec{x}_0\right]_B$$

$$= \sqrt{17}^t \begin{bmatrix} \frac{1}{2}\cos(\theta t) - \frac{3}{2}\sin(\theta t) & -\frac{1}{2}\sin(\theta t) - \frac{3}{2}\cos(\theta t) \\ -\sin(\theta t) & -\cos(\theta t) \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix}_B$$

$$= \sqrt{17}^t \begin{bmatrix} 5\sin(\theta t) \\ 3\sin(\theta t) + \cos(\theta t) \end{bmatrix}$$

Diagonalization and dynamical systems.

ex: Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find A^t and $\lim_{t\to\infty} A^t$

by graphing: $\lambda = 0, \pm 1$ $\quad\lambda=0 \quad \lambda=1 \quad \lambda=-1$

w/ corresponding eigenvectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$

$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$

w/ diagonal $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

ex: Find a closed form for $A^t \vec{x}_0$ and $\lim_{t \to \infty} A^t \vec{x}_0$

for $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 3 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$