Complex numbers.

$z = a + bi$

$|z|$ modulus.

θ argument of z

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$. (polar form of z).

<u>Thm</u>: De Moivre's Thm

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Rotations & Powers of z.

If $z = r(\cos\theta + i\sin\theta)$, then z^n spirals in/out as it rotates by θ about the unit circle.

<u>Thm</u>: <u>FToA</u>

Any poly $p(\lambda)$ w/complex coefficients splits, that is, it can be written as a product of linear factors.

$$p(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

for some complex numbers $\lambda_1, \dots, \lambda_n$, & k.

7.5
2/8

ex1: Show A is similar to a rotation-scaling matrix.

Given $A = \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}$

Given eigenvalues: $\lambda = 3+i$ and $\lambda = 3-i$

Given eigenvectors: $\vec{u}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

which may be written as

$$\vec{u}_1 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}} + \underbrace{i\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{i\vec{w}} \quad \text{and} \quad \vec{u}_2 = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}} - \underbrace{i\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{i\vec{w}}$$

We can verify:

$A = SBS^{-1}$ w/ $S = \begin{bmatrix} | & | \\ \vec{w} & \vec{v} \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Q: How do we find complex eigenvalues $\lambda = a \pm ib$?

Q: How do we find the corresponding eigenvectors $\vec{u} = \vec{v} + i\vec{w}$

Q: How do we know A is similar to $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

w/ $S = \begin{bmatrix} | & | \\ \vec{w} & \vec{v} \\ | & | \end{bmatrix}$?

$\boxed{\begin{matrix} 7.5 \\ 3/8 \end{matrix}}$

ex2: Diagonalize $A = \begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 11-\lambda & 6 \\ -15 & -7-\lambda \end{vmatrix}$$

$$= (11-\lambda)(-7-\lambda) + 90$$

$$= -77 - 4\lambda + \lambda^2 + 90$$

$$= \lambda^2 - 4\lambda + 13$$

solve $0 = \lambda^2 - 4\lambda + 13$. $\quad \lambda = \dfrac{4 \pm \sqrt{16 - 4(1)(13)}}{2(1)}$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

Find $\ker(\underbrace{A - (2+3i)I}$)

$$\begin{bmatrix} 9-3i & 6 \\ -15 & -9-3i \end{bmatrix} \quad \frac{1}{9-3i} R_1 \to R_1$$

$$\begin{bmatrix} 1 & \frac{3}{5} + \frac{1}{5}i \\ -15 & -9-3i \end{bmatrix} \quad 15R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & \frac{3}{5} + \frac{1}{5}i \\ 0 & 0 \end{bmatrix}$$

$$E_{2+3i} = \text{span}\Big(\overset{\vec{v}_1}{\begin{bmatrix} -3-i \\ 5 \end{bmatrix}} \Big)$$

and check if the conjugate of $\vec{v}_1$ is also an eigenvector. $\lambda_2 \vec{v}_2$

$$\begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix} \overset{\vec{v}_2}{\begin{bmatrix} -3+i \\ 5 \end{bmatrix}} = \begin{bmatrix} -3+11i \\ 10-15i \end{bmatrix}$$

So $\begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix} = P^{-1} A P$ (of the form $D = S^{-1}AS$)

where $P = \begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix}$.

Again $P^{-1} \begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix} P = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$ which is an example reminding us that if there is an eigenbasis for the transformation $T(\vec{x}) = A\vec{x}$ then A is diagonalizable.

In this case P & D contain complex entries.

Looking ahead, we just found $D = P^{-1} A P$

next we find $D = R^{-1} C R$

$\Rightarrow P^{-1} A P = R^{-1} C R$

$\Rightarrow A = P R^{-1} C R P^{-1}$

$\vdots$

but we're getting ahead of ourselves.

ex3: Diagonalize $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (rotation-scaling matrix)

Given: $\lambda = a \pm ib$ $\qquad$ $a, b \in \mathbb{R}$ and $b \neq 0$.

Given: $E_{a+ib} = \text{span} \begin{bmatrix} i \\ 1 \end{bmatrix}$

Given: $E_{a-ib} = \text{span} \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\Big\}\ R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

be careful – other eigenvectors don't work out. – see page 7. They must be written as complex conj.

$C = RDR^{-1}$ OR $D = R^{-1}CR$

and $R^{-1} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} R = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$

(of the form $S^{-1}AS = D$).

That is, we diagonalized the rotation-scaling matrix.

recall: IF A is a real 2×2 matrix w/ eigenvalues $a \pm ib$ $(b \neq 0)$ and corresponding eigenvectors $\vec{v} \pm \vec{w}i$, then

$$P^{-1}AP = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix} \text{ where } P = \begin{bmatrix} | & | \\ \vec{v}+\vec{w}i & \vec{v}-\vec{w}i \\ | & | \end{bmatrix}.$$

matrix w/ complex eigenvals.

rotation scaling matrix

$$\Rightarrow \underset{\text{matrix w/ complex eigenvals.}}{P^{-1}AP} = R^{-1} \underset{\text{rotation scaling matrix}}{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}} R \leftarrow P^{-1}AP = R^{-1}CR$$

$$\Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = RP^{-1}APR^{-1} = S^{-1}AS$$

$$\text{where } S = PR^{-1} = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

S has real values & so A is similar to a rotation-scaling matrix.

ex2 rev:

$$A = \begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix}$$

$$\lambda = 2 \pm 3i$$

$$E_{2+3i} = \text{span} \underbrace{\begin{bmatrix} -3-i \\ 5 \end{bmatrix}}_{\begin{bmatrix} -3 \\ 5 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$\vec{v} + i\vec{w}$$

$$\Rightarrow \quad S = \begin{bmatrix} -1 & -3 \\ 0 & 5 \end{bmatrix}$$

and $$S^{-1} A S = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $\lambda = a + ib$ is an eigenvalue.

what scaling factor?

what rotation?

Thm: A complex $n \times n$ matrix has n complex eigenvals if they are counted w/ alg. mult.

Thm: $\det(A) = \lambda_1 \cdot \ldots \cdot \lambda_n$

$\mathrm{Tr}(A) = \lambda_1 + \ldots + \lambda_n$.

THE FORMULA FOR S

If $R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix}$

Then $S = PR^{-1} = \frac{1}{2i}\begin{bmatrix} | & | \\ \vec{v}+i\vec{w} & \vec{v}-i\vec{w} \\ | & | \end{bmatrix}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$

$$= \frac{1}{2i}\begin{bmatrix} a+ib & a-ib \\ c+id & c-id \end{bmatrix}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} 2ib & 2ia \\ 2id & 2ic \end{bmatrix}$$

$$= \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \vec{w} & \vec{v} \\ | & | \end{bmatrix}$$

Thus we can find the change of basis matrix S w/o even knowing P or R. However, knowing P & remembering $R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ (always), we can verify our S.

CHANGE OF BASIS MATRICES REQUIRE COMPLEX CONJUGATE EIGENVECTORS

$\lambda = 2 \pm 3i$ for the matrices $A = \begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix}$ and $P = \begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix}$

Eigenvectors of $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ can take many forms. $\boxed{7.5 \; 8/8}$

$$\vec{V}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\vec{V}_2 = \underset{\vec{V}_1 \text{ conjugate}}{\begin{bmatrix} -i \\ 1 \end{bmatrix}} \text{ or } \underset{*(-1)}{\begin{bmatrix} i \\ -1 \end{bmatrix}} \text{ or } \underset{*(i)}{\begin{bmatrix} 1 \\ i \end{bmatrix}} \text{ or } \underset{*(2)}{\begin{bmatrix} -2i \\ 2 \end{bmatrix}}$$

which gives R many forms

$$R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \text{ or } \begin{bmatrix} i & -2i \\ 1 & 2 \end{bmatrix}$$

check R & P by comparing the product

$S = PR^{-1}$ to the formula for $S = [\vec{w} \;\; \vec{v}]$

where the eigenvectors of $A = \vec{V} + i\vec{w} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{V}_1$ conjugate : $PR^{-1} = \frac{1}{2i}\begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} -2i & -6i \\ 0 & 10i \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 0 & 5 \end{bmatrix} = S$

$*(-1)$: $PR^{-1} = -\frac{1}{2i}\begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix}\begin{bmatrix} -1 & -i \\ -1 & +i \end{bmatrix} = +\frac{1}{2i}\begin{bmatrix} -6 & +2 \\ +10 & 0 \end{bmatrix} = \begin{bmatrix} 3i & -i \\ -5i & 0 \end{bmatrix} \neq S$

$*(i)$: $PR^{-1} = -\frac{1}{2}\begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -4i+4 & 2-2i \\ 5i-5 & -5+5i \end{bmatrix} \neq S$

$*(2)$: $PR^{-1} = \frac{1}{4i}\begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2i \\ -1 & i \end{bmatrix} = \frac{1}{4i}\begin{bmatrix} -3-3i & -9i+1 \\ 5 & 15i \end{bmatrix} \neq S$