7.4: ① Dynamical systems.

vocab: $\vec{x}_0$ = initial state vector

$\vec{x}(t)$ = state vector

$x(t+1) = A\vec{x}(t)$

ex: road runners & coyotes (ex 7 in 7.1)

suppose $A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}$

and $\vec{x}_0 = \begin{bmatrix} c \\ r \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$

a) Find $\vec{x}(1)$ and $\vec{x}(10)$.

b) There are two eigenvals: 1.1 & 0.9

and $E_{1.1} = \text{span}\left(\begin{bmatrix} 100 \\ 300 \end{bmatrix}\right)$ and $E_{0.9} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

Diagonalize A and find a closed form for $\vec{x}(t)$.

$$\vec{x}(t) = A^t\vec{x}_0 = S\begin{bmatrix} 1.1^t & 0 \\ 0 & 0.9^t \end{bmatrix} S^{-1}\vec{x}_0 \quad w/\ S = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} 80\cdot(0.9^t) + 20(1.1^t) \\ 40(0.9^t) + 60(1.1^t) \end{bmatrix}$$

c) what happens as $t \to \infty$?

ex: consider A. Find A^t, $A^t\vec{x}_0$, and $\lim_{t\to\infty} A^t\vec{x}_0$

(a) $A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$

- A is a transition matrix as each col. sums to 1.
- List λ's in decreasing order.
- $\lambda = 1$ will appear for a transition matrix

$\lambda = 1$ OR $\lambda = \frac{1}{4}$

$E_1 = span(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix})$ and $E_{1/4} = span(\begin{bmatrix} -1 \\ 1 \end{bmatrix})$

so $A = SBS^{-1}$ w/ $S = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$

and $A^t = SB^tS^{-1}$

$= S \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4^t} \end{bmatrix} S^{-1}$

$= \frac{2}{3} \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4^t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1/2 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 1 + 2(0.25)^t & 1 - (0.25)^t \\ 2 - 2(0.25)^t & 2 + (0.25)^t \end{bmatrix}$

- Notice that $\lim_{t\to\infty} A^t = \begin{bmatrix} | & & | \\ \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & & | \end{bmatrix}$

(b) If $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $A^t\vec{x}_0$

$A^t\vec{x}_0 = \frac{1}{3} \begin{bmatrix} 1 + 2(0.25)^t & 1 - (0.25)^t \\ 2 - 2(0.25)^t & 2 + (0.25)^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 1 + 2(0.25)^t \\ 2 - 2(0.25)^t \end{bmatrix}$

(c) $\lim_{t\to\infty} A^t\vec{x}_0 = \frac{1}{3} \begin{bmatrix} 1 + 2(0.25)^t \\ 2 - 2(0.25)^t \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- This is in E_1 but has entries that sum to 1.

$= \vec{x}_{equ}$

Diagonalization Process. of $A_{n \times n}$

(a) Find the eigenvals.

(b) Find each eigenspace.

(c) if the sum of $\dim(E_\lambda) \neq n$, stop.

(d) else, construct D & S.

Thm: Powers of a Diagonalizable Matrix.

If A can be diagonalized as $A = SDS^{-1}$

Then $A^t = SD^tS^{-1}$.

Since A is assumed to be diagonalizable, there exists an eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$ for A, with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. We can order the eigenvectors so that $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j = 2, \ldots, n$. Now we can write

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Then

$$A^t\vec{x}_0 = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n = c_1\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n$$

and

$$\lim_{t\to\infty}(A^t\vec{x}_0) = \lim_{t\to\infty}(c_1\vec{v}_1 + \underbrace{c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n}_{\searrow\vec{0}}) = c_1\vec{v}_1.$$

ex: $A = \begin{bmatrix} 0 & 0.5 & 0.4 \\ 1 & 0 & 0.6 \\ 0 & 0.5 & 0 \end{bmatrix}$

$$\begin{array}{ccc} \text{1st} & \text{3rd} & \text{2nd} \\ \lambda = 1, & -0.2763932, & -0.7236068 \\ \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 1.4 \\ 2 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.4472 \\ -0.5528 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.4472 \\ -1.4472 \\ 1 \end{bmatrix} \end{array}$$

order by decreasing magnitude.

$$\Rightarrow A = S B S^{-1} \quad w/ \quad S = \begin{bmatrix} 1.4 & 0.4472 & -0.4472 \\ 2 & -1.4472 & -0.5528 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.7236 & 0 \\ 0 & 0 & -0.2764 \end{bmatrix}$$

$$\Rightarrow A^t = \begin{bmatrix} .32 - .29|\lambda_2|^t - .39|\lambda_3|^t & .32 + .21|\lambda_2|^t + .11|\lambda_3|^t & .32 - .01|\lambda_2|^t + .33|\lambda_3|^t \\ .45 + .94|\lambda_2|^t - .48|\lambda_3|^t & .45 - .68|\lambda_2|^t + .13|\lambda_3|^t & .45 + .04|\lambda_2|^t + .41|\lambda_3|^t \\ .23 - .65|\lambda_2|^t + .88|\lambda_3|^t & .23 + .47|\lambda_2|^t - .24|\lambda_3|^t & .23 - .03|\lambda_2|^t - .74|\lambda_3|^t \end{bmatrix}$$

$$\text{and} \quad \lim_{t\to\infty} A^t = \frac{1}{22} \begin{bmatrix} 7 & 7 & 7 \\ 10 & 10 & 10 \\ 5 & 5 & 5 \end{bmatrix}$$

$$\text{and} \quad \lim_{t\to\infty} A^t \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$