7.2: The eigenvalue problem.

Dfn: For an $n \times n$ matrix A, find all scalars λ s.t $A\vec{v} = \lambda\vec{v}$ has a non-zero solution $\vec{v}$. Such a scalar λ is an eigenvalue w/ corresponding eigenvector $\vec{v}$.

Q: What do eigenvectors/values do geometrically?

So how do we find them?

solve $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq \vec{0}$

$\Rightarrow \quad Av - \lambda v = \vec{0}$

$\Rightarrow \quad (A - \lambda I)\vec{v} = \vec{0}$

If this is to have nontrivial solutions then $A - \lambda I$ must be singular

Step 1: Find all scalars λ s.t. $A - \lambda I$ is singular ... that is $\det(A - \lambda I) = 0$.

Step 2: Given a scalar λ s.t. $A - \lambda I$ is singular, find all nonzero $\vec{v}$'s s.t. $(A - \lambda I)\vec{v} = \vec{0}$.

ex1: Find the eigenvalues of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Soln: characteristic poly $\lambda^2 - 4\lambda + 4$

$\lambda = 2$ (alg. mult 2).

If A $(n \times n)$, then $\det(A - \lambda I)$ is a poly w/ degree n. We call it the characteristic poly.

The zeros of the char. poly are the eigenvals.

ex2: Find the eigenvals of $B = \begin{array}{c} \begin{matrix} -\frac{1}{2} & 1 & 0 \end{matrix} \\ \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix} \end{array}$

$\lambda = 0$; $\lambda = -1$ (alg. mult. 2)

By FToA

(a) an $n \times n$ matrix can have no more than n distinct eigen vals.

(b) an $n \times n$ matrix always has @ least one eigenval. (possibly complex)

(c) an $n \times n$ matrix w/ n odd has at least one real eigenval.

Thm: Let A be an $n \times n$ matrix w/ eigenvalue λ

requires induction: give handout

prove it. → (a) λ^k is an eigenval for A^k, $k = 2, 3, \ldots$

→ (b) If A is nonsingular, $\frac{1}{\lambda}$ is an eigenval. of A^{-1}.

(c) If α is a scalar, then $(\lambda + \alpha)$ is an eigenval of $(A + \alpha I)$.

Thm: Let A be an $n \times n$ matrix. Then A and A^T have the same eigenvals.

Thm: Let A be an $n \times n$ matrix. Then A is singular iff $\lambda = 0$ is an eigenval.

Thm: Let T be an $n \times n$ triangular matrix. Then the eigenvals of T are its diag. entries.

<u>claim</u>: If $A_{n\times n}$ has eigenvalue λ, then λ^k is an eigenvalue for A^k for $k=2,3,\dots$

□ <u>proof by mathematical induction.</u>

λ is an eigenvalue for A. This means there is a nonzero $\vec{v}$ s.t. $A\vec{v}=\lambda\vec{v}$.

$P(1)$ $\Rightarrow$ $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$

so λ^2 is an eigenvalue of A^2.

& $P(k)$ assume λ^k is an eigenvalue of A^k

then $P(k+1)$ $\Rightarrow$ $A^{k+1}\vec{v} = A(A^k\vec{v}) = A(\lambda^k\vec{v}) = \lambda^k A\vec{v} = \lambda^{k+1}\vec{v}$

so λ^{k+1} is an eigenvalue of A^{k+1}

Hence our claim is proved by mathematical induction. ■

<u>claim</u>: If $A_{n\times n}$ is nonsingular w/ eigenvalue λ then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

□ <u>proof</u>.

λ is an eigenvalue of A. So there is $\vec{v}\neq\vec{0}$ s.t. $A\vec{v}=\lambda\vec{v}$.

$\Rightarrow$ $A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v})$

$\Rightarrow$ $\vec{v} = \lambda A^{-1}\vec{v}$

$\Rightarrow$ $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$

Hence $\frac{1}{\lambda}$ is an eigenval. of A^{-1} ■