6.1 : Determinants

The 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $\det A = ad-bc \neq 0$.

More generally, the $n\times n$ matrix A is invertible if $\det A \neq 0$. ... But how do you find $\det A$?

In the 3×3 case, we need the $\text{im}(A)=\mathbb{R}^3$ or the cols to be L.I. One way to determine this is as follows. Let $A=[\vec{u}\ \vec{v}\ \vec{w}]$

$$\det A = \vec{u}\cdot(\vec{v}\times\vec{w}).$$

Q: why?

ex1: calculate $\det A$ for $A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 4 & 5 & 1 \end{bmatrix}$

using dfn & Sarrus Rule.

This is called the Laplace Cofactor expansion,

Det. of a triangular or diag. matrix is simply the product of the diag ex2: Find the det A for $A = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 3 \\ -2 & 1 & 1-\lambda \end{bmatrix}$.

what does this mean? key: $\lambda = 0, 2, 3$

Note: In the 3×3 case, $\det B = -\det A$ if A & B differ only in a single col/row swap.

evaluating determinates ... the signs.

ex3: $A = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}$

* NO change to the det if rows added.
* scaling changes the det by the scalar.

$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 4 & 2 & 2 \end{bmatrix}$

row & col swaps.

To see how it works, work w/ $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$.

$C = \begin{bmatrix} 0 & 4 & 1 & 3 \\ 0 & 2 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 4 \end{bmatrix}$

Determinants Using Row Operations

$$|A| = \begin{vmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \quad R_1 \leftrightarrow R_2$$

Flops by algorithm

① Laplace cofactor expansions: $O(n!)$

② row/col ops: $O(n^3)$

$$= -\begin{vmatrix} 3 & 0 & -3 & 6 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \quad \tfrac{1}{3} R_1 \mapsto R_1$$

$$= -3\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \quad \begin{matrix} R_3 - 2R_1 \to R_3 \\ R_4 - 5R_1 \to R_4 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 4 & 7 & 3 \\ 0 & -1 & 2 & -9 \end{vmatrix} \quad R_2 \leftrightarrow R_4$$

$$= -(-3)\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{vmatrix} \quad \begin{matrix} R_3 + 4R_2 \to R_3 \\ R_4 + 2R_2 \to R_4 \end{matrix}$$

$$= 3\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 0 & 15 & -33 \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

$$= 3(1)(-1)(15)(-13)$$

$$= 585$$