

3.3: Dimension of a Subspace of \mathbb{R}^N

Intuitively, 2 vectors in \mathbb{R}^3 determine a plane (non-zero, not parallel). Thus we might guess (accurately) that the dimension of the plane is 2...

Recall: $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^N$ are a basis for a subspace V provided they span and are L.I.



Thm: Consider vectors $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}_1, \dots, \vec{w}_q$ in a subspace V of \mathbb{R}^N . If $\vec{v}_1, \dots, \vec{v}_p$ are L.I. and $\vec{w}_1, \dots, \vec{w}_q$ span V , then $q \geq p$.

Thm: All bases of a subspace V of \mathbb{R}^N have the same number of vectors.

□ proof.

Consider 2 bases for V : $\vec{v}_1, \dots, \vec{v}_p$ & $\vec{w}_1, \dots, \vec{w}_q$.

$\vec{v}_1, \dots, \vec{v}_p$ are LI & $\vec{w}_1, \dots, \vec{w}_q$ span V so $p \leq q$

& $\vec{w}_1, \dots, \vec{w}_q$ are LI & $\vec{v}_1, \dots, \vec{v}_p$ span V so $q \leq p$

$\Rightarrow p = q$ QED. ■

Think back to the basis for our plane...

Defn. Consider a subspace V of \mathbb{R}^N . The number of vectors in a basis of V is called the dimension of V and denoted by $\dim(V)$.

Thm. Consider a subspace V of \mathbb{R}^N w/ $\dim(V) = m$.

- (a) We can find at most m L.I. vectors in V .
- (b) We need at least m vectors to span V .
- (c) ~~the~~ If m vecs in V are L.I., then they form a basis for V .
- (d) If m vecs span V , then they form a basis for V .

□ proof of (d).

Assume $\text{span}(\vec{v}_1, \dots, \vec{v}_m) = V$. NTS $\vec{v}_1, \dots, \vec{v}_m$ are L.I.

Suppose $\vec{v}_1, \dots, \vec{v}_m$ are L.D.

$\Rightarrow \vec{v}_i = c_1 \vec{v}_1 + \dots + c_{i-1} \vec{v}_{i-1}$ (\vec{v}_i is redundant).

$\Rightarrow \vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$ spans V

$\Rightarrow (m-1)$ vectors span V .

\Rightarrow (b) is false. $\Rightarrow \Leftarrow$

$\therefore \vec{v}_1, \dots, \vec{v}_m$ are L.I.

ex1: Find a basis for the kernel & image of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}_{5 \times 4}$$

(a) To find the kernel, solve $A\vec{x} = \vec{0}$

$$\text{rref}(A|\vec{0}) = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{so } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ 0 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

All vectors in $\ker(A)$ are scalar multiples of $\vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$
 so \vec{w}_1 forms a basis for $\ker(A)$.

(b) To find the image, we must find all lin. comb. of the columns of A . ~~which columns~~ How many cols are redundant?

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

← one row of zeros \Rightarrow one ~~row~~ redundant column.

better yet, the non-zero cols of $[rref(AT)]^T$ form a basis for $Im(A)$.

$$\text{so } \vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

are a basis...

An alternate method... $rref(A)$ and identify the cols. w/ a leading 1. These cols in A form a basis for $Im(A)$.

$$\text{so } \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}; \vec{b}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}; \vec{b}_4 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

are a basis for $Im(A)$.

Think about it this way, each col w/ a leading 1 in $\text{rref}(A)$ corresponds to basis vec. in $\text{im}(A)$. The other cols give you free variables, and basis vecs for $\text{ker}(A)$.

$$m = \underbrace{\dim(\text{im}(A))}_{\text{rank}(A)} + \underbrace{\dim(\text{ker}(A))}_{\text{nullity}(A)}$$

So $T(\vec{x}) = A\vec{x}$ where $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and A $n \times m$ sends $\text{nullity}(A)$ cols to the $\text{kernel}/\vec{0}$ & the $\text{rank}(A)$ cols to the image & the lin. comb. of the remaining form the image.

ex2: Find a basis for $\text{im}(A)$ & $\text{ker}(A)$ of

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix} \Rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{im}(A) = \text{span} \left(\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right)$$

$$\text{ker}(A) = \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Thm: $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ are a basis for \mathbb{R}^n iff $\begin{bmatrix} | & & | \\ \vec{v}_1 & & \vec{v}_n \\ | & & | \end{bmatrix}$ is invertible.