basis $\mathcal{A}$. Now $\begin{bmatrix} 1 \\ a \\ a^2 \\ a^3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + a \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \\ b^2 \\ b^3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, so that $S = S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}$

g. Using the equations $1 + a = a^2$ and $1 + b = b^2$, we find that $AS = SB = \begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix}$.

4.3.73 a. To check orthogonality, verify that $\vec{x} \cdot T(\vec{x}) = 0$. To check that $T(\vec{x}) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{bmatrix}$ is in V if $\vec{x}$ is in V, we need to verify that $y_3 = y_1 + y_2$ and $y_4 = y_2 + y_3$, meaning that $x_2 = x_4 - x_3$ and $-x_1 = -x_3 + x_2$. But the two last equations follow from the definition of V.

b. $F\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $F\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, so that $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

c. $F\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $F\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, so that $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

d. To write the change of basis matrix $S_{\mathcal{B}\to\mathcal{A}}$, we need to express the vectors of basis $\mathcal{B}$ in terms of the vectors of basis $\mathcal{A}$. Now $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, so that $S = S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

e. We find that $AS = SB = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$.

f. No such basis $\mathcal{C}$ exists, since the rotation matrix $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ from part c fails to be similar to a diagonal matrix, by Example 3.4.10.

True or False

Ch 4.TF.1 T; We are looking at P_6, with a basis $1, t, t^2, t^3, t^4, t^5, t^6$, which has seven elements.

Ch 4.TF.2 T; We can check both requirements of Definition 4.2.1.

Ch 4.TF.3 T; check the three properties listed in Definition 4.1.2.

Ch 4.TF.4 T; by Definition 4.2.1.

Ch 4.TF.5 F; A basis of $\mathbb{R}^{2\times 3}$ is $\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},$

$\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}$, so it has a dimension of 6.

Ch 4.TF.6 T; check with Definition 4.1.3c.

Ch 4.TF.7 T; The linear transformation $T(ax+b) = a+ib$ is an isomorphism from P_1 to $\mathbb{C}$, with the inverse $T^{-1}(a+ib) = ax+b$.

Ch 4.TF.8 T; by Theorem 4.2.4c.

Ch 4.TF.9 T; This fits all properties of Definition 4.1.2.

Ch 4.TF.10 F; The transformation T could be: $T(f) = \begin{bmatrix}0&0\\0&0\end{bmatrix}$, in which case the kernel would be all of P_6 and the dimension of the kernel would be 7.

Ch 4.TF.11 F; $t^3, t^3+t^2, t^3+t, t^3+1$ is a basis of P_3.

Ch 4.TF.12 T; If T is linear and invertible, then T^{-1} will be linear and invertible as well.

Ch 4.TF.13 F; $T(\sin(x)) = \sin(x) - \sin(x) = 0$.

Ch 4.TF.14 F; $T(f) = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not an isomorphism.

Ch 4.TF.15 F; Let V=$\mathbb{R}^2$, $A = \begin{bmatrix}1&-1\\1&-1\end{bmatrix}$. Now $\text{im}(A) = \ker(A) = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$.

Ch 4.TF.16 T; the dimensions of both spaces are the same: 10.

Ch 4.TF.17 F; $\dim(P_3)$= 4, so the three given polynomials cannot span P_3.

Ch 4.TF.18 T; We can construct a basis of V by omitting the redundant elements from a list of ten elements that span V. Thus $\dim(V) \le 10$.

Ch 4.TF.19 F; $\det\begin{bmatrix}1&0\\0&0\end{bmatrix} + \det\begin{bmatrix}0&0\\0&1\end{bmatrix} = 0 \ne \det\begin{bmatrix}1&0\\0&1\end{bmatrix} = 1$.

Ch 4.TF.20 F; For any matrix A, the space of matrices commuting with A is at least two-dimensional. Indeed, if A is a scalar multiple of I_2, then A commutes with all 2×2 matrices, and if A fails to be a scalar multiple of I_2, then A commutes with the linearly independent matrices A and I_2.

Ch 4.TF.21 F; $T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = \begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a+2c&b+2d\\3a+6c&3b+6d\end{bmatrix}$

$= (a+2c)\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix} + (b+2d)\begin{bmatrix}0 & 1\\ 0 & 3\end{bmatrix}$. So the image is the span of $\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 0 & 3\end{bmatrix}$, and rank($T$)= 2.

Ch 4.TF.**22** T; If the basis $\mathcal{B}$ we consider is f_1, f_2, then the given matrix tells us that $T(f_1) = 3f_1$ and $T(f_2) = 5f_1 + 4f_2$. Thus $f = f_1$ does the job.

Ch 4.TF.**23** T; If $T(f(t)) = f(t^2) = 0$, $f(t)$ must also be zero.

Ch 4.TF.**24** T; The inverse is $T^{-1}(N) = S^{-1}NS^{-1}$. .

Ch 4.TF.**25** T; Let $A = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$. Then we want $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, or

$\begin{bmatrix}a+c & b+d\\ 0 & 0\end{bmatrix} = \begin{bmatrix}a & a\\ c & c\end{bmatrix}$. Thus, $c = 0$ and $a = b + d$. So our space is the span of $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}$.

Ch 4.TF.**26** T; Let our basis be $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$. Each matrix here is invertible, and also clearly none are redundant.

Ch 4.TF.**27** F; $T(f(t)) = f'(t)$ is not an isomorphism.

Ch 4.TF.**28** T; We need only show that either the new list contains no redundant elements, or spans the whole space. The latter is slightly easier to show. Since f_1, f_2, f_3 form a basis of V, it suffices to show that these three elements are in the span of $f_1, f_1 + f_2, f_1 + f_2 + f_3$. This is simple to demonstrate: $f_2 = (f_1 + f_2) - f_1$, and $f_3 = (f_1 + f_2 + f_3) - (f_1 + f_2)$.

Ch 4.TF.**29** T; We show that none of the polynomials is redundant; let's call them $f(x)$, $g(x)$ and $h(x)$. Now $g(x)$ isn't a multiple of $f(x)$ since $f(b) = 0$, but $g(b) \neq 0$. Likewise, $h(x)$ isn't a linear combination of $f(x)$ and $g(x)$ since $f(c) = g(c) = 0$, but $h(c) \neq 0$.

Ch 4.TF.**30** T; Make the substitution $4t - 3 = s$ to see that the inverse is $T^{-1}(g(s)) = g(\frac{s+3}{4})$.

Ch 4.TF.**31** F; P_2 is a subspace of P, and P is infinite dimensional.

Ch 4.TF.**32** T; Let $T\left(\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\right) = \begin{bmatrix}b & c\\ 0 & f\end{bmatrix}$. We can easily see that the kernel and image of this transformation are exactly as required.

Ch 4.TF.**33** F; The space spanned by $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ contains no invertible matrices.

Ch 4.TF.**34** F; This is the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$. The change of basis matrix we are looking for is: $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$.

Ch 4.TF.**35** F; Let $\mathcal{B} = (f, g)$ and $\mathcal{C} = (g, f)$. The fact that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the $\mathcal{B}$- matrix of T implies that $[T(f)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, or $T(f) = f + 3g$. But then $[T(f)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, meaning that the second column of the $\mathcal{C}$-matrix of T is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This shows that the matrix $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ fails to be the $\mathcal{C}$-matrix of T.

Ch 4.TF.**36** T; The image of T is P_{n-1}, so that $\text{rank}(T) = \dim(\text{im}T) = \dim(P_{n-1}) = n$.

Ch 4.TF.**37** T; because the matrix is invertible.

Ch 4.TF.**38** T; The dimension of P_9 is 10, and the dimension of $\mathbb{R}^{3\times 4}$ is 12. Thus, any 10-dimensional subspace of $\mathbb{R}^{3\times 4}$ will be acceptable. For example, we can consider the space of all 3×4 matrices A with $a_{11} = a_{12} = 0$.

Ch 4.TF.**39** T; let W_1 be $\{\vec{0}\}$. Then any other subspace W_2 unioned with W_1 will simply be W_2 again, which we know is a subspace.

Ch 4.TF.**40** T; Let $T(a_0 + a_1t + a_2t^2 + \cdots + a_5t^5 + \cdots) = a_0 + a_1t + a_2t^2 + \cdots + a_5t^5$. The image of this transformation is clearly all of P_5, and T satisfies the requirements of Definition 4.2.1.

Ch 4.TF.**41** T; there will be no redundant elements in this list.

Ch 4.TF.**42** F; The kernel of T consists of all constant functions.

Ch 4.TF.**43** T; We apply the rank-nullity theorem: $\dim(W) = \dim(\text{im}(T)) = \dim(P_4) - \dim(\ker(T)) = 5 - \dim(\ker(T)) \leq 5$.

Ch 4.TF.**44** F; We can construct as many linearly independent elements in $\ker(T)$ as we want, for example, the polynomials $f(t) = t^n - \frac{1}{n+1}$, for all positive integers n.

Ch 4.TF.**45** T; 0 is in our set, and if f and g are in our set, then $T(f+g) = T(f) + T(g) = f + g$ so that $f + g$ is in our set as well. Also, if f is in our set and k is an arbitrary scalar, then $T(kf) = kT(f) = kf$, so kf is in our set as well.

Ch 4.TF.**46** T; The kernel of T is $\{0\}$. Indeed, if $f(t)$ is a nonzero polynomial, with $f(t) = a_0 + a_1t + ... + a_kt^k$ where $a_k \neq 0$, then $T(f(t)) = a_0T(1) + a_1T(t) + ... + a_kT(t^k)$is of degree $k \geq 0$, so that $T(f(t))$fails to be the zero polynomial.

Ch 4.TF.**47** T; Let $P = I_2$, $Q = -I_2$. Then $T(M) = I_2M - M(-I_2) = 2M$, which is an isomorphism.

Ch 4.TF.**48** F; We use dimension arithmetic here to show that this cannot happen. Any transformation T from P_6 to $\mathbb{C}$ must have a kernel of at least 5 dimensions, since P_6 is 7-dimensional and $\mathbb{C}$ is only a 2-dimensional space. Thus, any such kernel cannot be isomorphic to $\mathbb{R}^{2\times 2}$, which is a 4-dimensional space.

Ch 4.TF.**49** F; If $f = -f_1$, then 0 is a member of the list!

Ch 4.TF.**50** T; Consider the space of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, for example.

Ch 4.TF.**51** T; note that $\dim(P_{11}) = 12 = \dim(\mathbb{R}^{3\times4})$. The linear spaces P_{11} and $\mathbb{R}^{3\times4}$ are both isomorphic to $\mathbb{R}^{12}$, via the coordinate transformation, and thus they are isomorphic to each other.

Ch 4.TF.**52** F; Consider the linear transformation $T(f(t)) = f(t)$ from P_2 to P, for example.

Ch 4.TF.**53** T; We use the rank-nullity theorem: $\dim(V) = \dim(\text{im}(T)) + \dim(\ker(T)) = \dim(\text{im}(T)) \leq \dim(\mathbb{R}^{2\times2}) = 4$.

Ch 4.TF.**54** T; Using the fundamental theorem of calculus, we can write $g(t) = T(f(t)) = 3f(3t+4)$. Make the substitution $3t + 4 = s$ to see that the inverse is $T^{-1}(g(s)) = g((s-4)/3)/3$.

Ch 4.TF.**55** T; Using a coordinate transformation, it suffices to show this for $\mathbb{R}^4$. For every real number k, we define the three dimensional subspace V_k of $\mathbb{R}^4$ consisting of all vectors $\vec{x}$ such that $x_4 = kx_3$. If c is different from k, then V_c and V_k will be different subspaces of $\mathbb{R}^4$, since V_k contains the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \\ k \end{bmatrix}$, but V_c does not. Thus we have generated infinitely many distinct three-dimensional subspaces V_k of $\mathbb{R}^4$, one for every real number k.

Ch 4.TF.**56** T; If the basis $\mathcal{B}$ we consider is f_1, f_2, then the given matrix tells us that $T(f_1) = 3f_1$ and $T(f_2) = 5f_1 + 4f_2$. We are looking for a nonzero $f = af_1 + bf_2$ such that $T(f) = 4f$. Now $T(f) = aT(f_1) + bT(f_2) = 3af_1 + 5bf_1 + 4bf_2 = (3a+5b)f_1 + 4bf_2$ must be equal to $4f = 4af_1 + 4bf_2$. Thus it is required that $3a + 5b = 4a$, or $a = 5b$. For example, $f = 5f_1 + f_2$ does the job.

Ch 4.TF.**57** T; This is logically equivalent to the following statement: If the domain of T is finite dimensional, then so is the image of T. Compare with Exercises 4.2.81a and 4.1.57.

Ch 4.TF.**58** F; If A is a scalar multiple of I_2, then all 2×2 matrices commute with A, so that the space of commuting matrices is 4 - dimensional. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ fails to be a scalar multiple of I_2, consider the equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which amounts to the system $cy - bz = 0, bx + (d-a)y - bt = 0, cx + (d-a)z - ct = 0$. If $b \neq 0$, then the first two equations are independent; if $c \neq 0$, then the first and the third equation are independent; and if $a \neq d$, then the second and the third equation are independent. Thus the rank of the system is at least two and the solution space is at most two-dimensional. (The solution space is in fact two -dimensional, since A and I_2 are independent solutions.)

Ch 4.TF.**59** T; If $A = 0$, then we are done. If $\text{rank}(A) = 1$, then the image of the linear transformation $T(M) = AM$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$ is two dimensional (if $\vec{v}$ is a basis of $\text{im}(A)$, then $\begin{bmatrix} \vec{v} & \vec{0} \end{bmatrix}, \begin{bmatrix} \vec{0} & \vec{v} \end{bmatrix}$ is a basis of $\text{im}(T)$). Since the three matrices $AB = T(B)$, $AC = T(C)$, and $AD = T(D)$ are all in $\text{im}(T)$, they must be linearly dependent.

Ch 4.TF.**60** F; Consider two distinct three-dimensional subspaces W_1 and W_2 of P_4. Since the spaces W_1 and W_2 are distinct, neither of them is a subspace of the other, so that we can find a polynomial f_1 that is in W_1 but not in W_2 as well as an f_2 that is in W_2 but not in W_1. Then f_1 and f_2 are both in the union of W_1 and W_2, but $f_1 + f_2$ isn't.

Ch 4.TF.**61** T; Pick the *first* redundant element f_k in the list. Since the elements $f_1, \ldots, f_{k-1}$ are linearly independent, the representation of f_k as a linear combination of the preceding elements will be unique.

Ch 4.TF.**62** F; $T(I_3) = P - P = 0$, and T can never be an isomorphism.

Ch 4.TF.**63** T; Let $W = \text{span}(f_1, f_2, f_3, f_4, f_5) = \text{span}(f_2, f_4, f_5, f_1, f_3)$. If we omit the two redundant elements from the first list, f_1, f_2, f_3, f_4, f_5, we end up with a basis of W with three elements, so that $\dim(W) = 3$. If we omit the redundant elements from the second list, f_2, f_4, f_5, f_1, f_3, we end up with a (possibly different) basis of W, but that basis must consist of 3 elements as well. Thus there must be two redundant elements in the second list.

Ch 4.TF.**64** F; The dimensions of the kernel and image would have to be equal, and both add up to the dimension of P_6, which is the odd number 7.

Ch 4.TF.**65** T; Consider the proof of the rank nullity theorem outlined in Exercise 4.2.81. In the proof, we use bases of $\ker(T)$ and $\text{im}(T)$ to construct a basis of the domain.

Ch 4.TF.**66** F; If the basis $\mathcal{B}$ we consider is f_1, f_2, then the given matrix tells us that $T(f_1) = 3f_1$ and $T(f_2) = 5f_1 + 4f_2$. We are looking for a nonzero $f = af_1 + bf_2$ such that $T(f) = 5f$. Now $T(f) = aT(f_1) + bT(f_2) = 3af_1 + 5bf_1 + 4bf_2 = (3a + 5b)f_1 + 4bf_2$ must be equal to $5f = 5af_1 + 5bf_2$. Thus it is required that $3a + 5b = 5a$ and $4b = 5b$, implying that $a = b = 0$. We are unable to find a nonzero f with the desired property.

Ch 4.TF.**67** T; Consider a 3-dimensional subspace W of $\mathbb{R}^{2\times 2}$. The matrices $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ in W can be described by a single linear equation $ax + by + cz + dt = 0$, where at least one of the coefficients is nonzero. Suppose x is the leading variable (meaning that $a \neq 0$), and y, z and t are the free variables. We can choose $y = z = 1$ and $t = 0$, and the resulting matrix $\begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix}$ in W will be invertible. We represent x by a star, since its value does not affect the invertibility. If y is the leading variable and the other three are the free variables, then we can construct the invertible matrix $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ in W. If z is the leading variable, we have the invertible matrix $\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$. Finally, for the leading variable t we have $\begin{bmatrix} 0 & 1 \\ 1 & * \end{bmatrix}$.