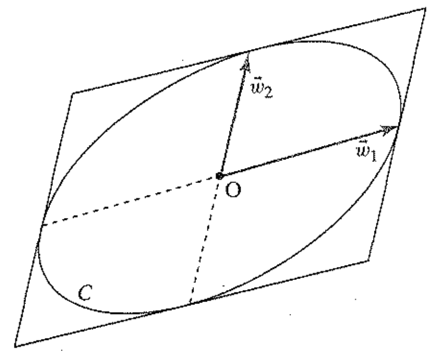
50. Let T be an invertible linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that the image of the unit circle is an ellipse centered at the origin.[8] [*Hint*: Consider two perpendicular unit vectors $\vec{v}_1$ and $\vec{v}_2$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular.] (See Exercise 47d.) The unit circle consists of all vectors of the form

$$\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2,$$

where t is a parameter.

51. Let $\vec{w}_1$ and $\vec{w}_2$ be two nonparallel vectors in $\mathbb{R}^2$. Consider the curve C in $\mathbb{R}^2$ that consists of all vectors of the form $\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$, where t is a parameter. Show that C is an ellipse. (*Hint:* You can interpret C as the image of the unit circle under a suitable linear transformation; then use Exercise 50.)



52. Consider an invertible linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let C be an ellipse in $\mathbb{R}^2$. Show that the image of C under T is an ellipse as well. (*Hint*: Use the result of Exercise 51.)

2.3 Matrix Products

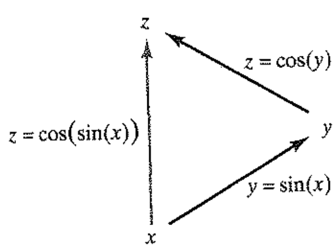


Figure 1

Recall the *composition* of two functions: The composite of the functions $y = \sin(x)$ and $z = \cos(y)$ is $z = \cos(\sin(x))$, as illustrated in Figure 1.

Similarly, we can compose two linear transformations.

To understand this concept, let's return to the coding example discussed in Section 2.1. Recall that the position $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of your boat is encoded and that you radio the encoded position $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ to Marseille. The coding transformation is

$$\vec{y} = A\vec{x}, \quad \text{with} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

In Section 2.1, we left out one detail: Your position is radioed on to Paris, as you would expect in a centrally governed country such as France. Before broadcasting to Paris, the position $\vec{y}$ is again encoded, using the linear transformation

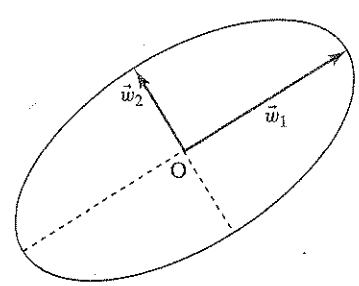
$$\vec{z} = B\vec{y}, \quad \text{with} \quad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

[8] An ellipse in $\mathbb{R}^2$ centered at the origin may be defined as a curve that can be parametrized as

$$\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2,$$

for two perpendicular vectors $\vec{w}_1$ and $\vec{w}_2$. Suppose the length of $\vec{w}_1$ exceeds the length of $\vec{w}_2$. Then we call the vectors $\pm\vec{w}_1$ the semimajor axes of the ellipse and $\pm\vec{w}_2$ the semiminor axes.

Convention: All ellipses considered in this text are centered at the origin unless stated otherwise.



this time, and the sailor in Marseille radios the encoded position $\vec{z}$ to Paris. (See Figure 2.)

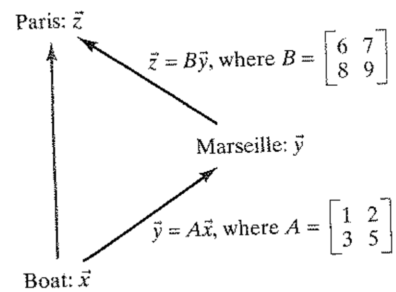


Figure 2

We can think of the message $\vec{z}$ received in Paris as a function of the actual position $\vec{x}$ of the boat,

$$\vec{z} = B(A\vec{x}),$$

the composite of the two transformations $\vec{y} = A\vec{x}$ and $\vec{z} = B\vec{y}$. Is this transformation $\vec{z} = T(\vec{x})$ linear, and, if so, what is its matrix? We will show two approaches to these important questions: (a) using brute force, and (b) using some theory.

a. We write the components of the two transformations and substitute.

$$\begin{aligned} z_1 &= 6y_1 + 7y_2 \\ z_2 &= 8y_1 + 9y_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y_1 &= x_1 + 2x_2 \\ y_2 &= 3x_1 + 5x_2 \end{aligned}$$

so that

$$\begin{aligned} z_1 &= 6(x_1 + 2x_2) + 7(3x_1 + 5x_2) = (6 \cdot 1 + 7 \cdot 3)x_1 + (6 \cdot 2 + 7 \cdot 5)x_2 \\ &= 27x_1 + 47x_2, \\ z_2 &= 8(x_1 + 2x_2) + 9(3x_1 + 5x_2) = (8 \cdot 1 + 9 \cdot 3)x_1 + (8 \cdot 2 + 9 \cdot 5)x_2 \\ &= 35x_1 + 61x_2. \end{aligned}$$

This shows that the composite is indeed linear, with matrix

$$\begin{bmatrix} 6 \cdot 1 + 7 \cdot 3 & 6 \cdot 2 + 7 \cdot 5 \\ 8 \cdot 1 + 9 \cdot 3 & 8 \cdot 2 + 9 \cdot 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}.$$

b. We can use Theorem 1.3.10 to show that the transformation $T(\vec{x}) = B(A\vec{x})$ is linear:

$$\begin{aligned} T(\vec{v} + \vec{w}) &= B\big(A(\vec{v} + \vec{w})\big) = B(A\vec{v} + A\vec{w}) \\ &= B(A\vec{v}) + B(A\vec{w}) = T(\vec{v}) + T(\vec{w}), \\ T(k\vec{v}) &= B\big(A(k\vec{v})\big) = B\big(k(A\vec{v})\big) = k\big(B(A\vec{v})\big) = kT(\vec{v}). \end{aligned}$$

Once we know that T is linear, we can find its matrix by computing the vectors $T(\vec{e}_1) = B(A\vec{e}_1)$ and $T(\vec{e}_2) = B(A\vec{e}_2)$; the matrix of T is then $\big[T(\vec{e}_1) \quad T(\vec{e}_2)\big]$, by Theorem 2.1.2:

$$T(\vec{e}_1) = B(A\vec{e}_1) = B(\text{first column of } A) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 27 \\ 35 \end{bmatrix},$$

$$T(\vec{e}_2) = B(A\vec{e}_2) = B(\text{second column of } A) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ 61 \end{bmatrix}.$$

We find that the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$ is

$$\begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}.$$

This result agrees with the result in (a), of course.

The matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$ is called the *product* of the matrices B and A, written as BA. This means that

$$T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x},$$

for all vectors $\vec{x}$ in $\mathbb{R}^2$. (See Figure 3.)

Now let's look at the product of larger matrices. Let B be an $n \times p$ matrix and A a $p \times m$ matrix. These matrices represent linear transformations as shown in Figure 4.

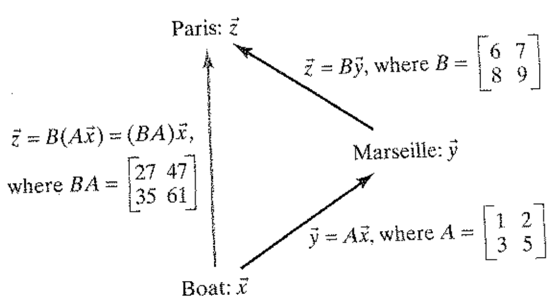


Figure 3

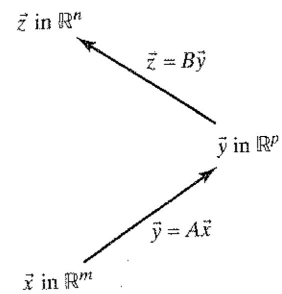


Figure 4

Again, the composite transformation $\vec{z} = B(A\vec{x})$ is linear. (The foregoing justification applies in this more general case as well.) The matrix of the linear transformation $\vec{z} = B(A\vec{x})$ is called the *product* of the matrices B and A, written as BA. Note that BA is an $n \times m$ matrix (as it represents a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$). As in the case of $\mathbb{R}^2$, the equation

$$\vec{z} = B(A\vec{x}) = (BA)\vec{x}$$

holds for all vectors $\vec{x}$ in $\mathbb{R}^m$, by definition of the product BA. (See Figure 5.)

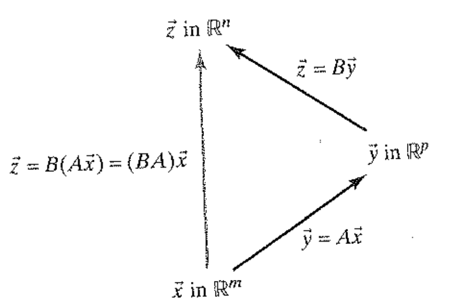


Figure 5

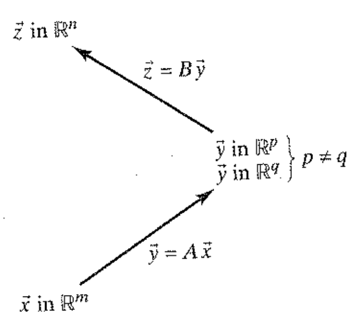


Figure 6

In the definition of the matrix product BA, the number of columns of B matches the number of rows of A. What happens if these two numbers are different? Suppose B is an $n \times p$ matrix and A is a $q \times m$ matrix, with $p \neq q$.

In this case, the transformations $\vec{z} = B\vec{y}$ and $\vec{y} = A\vec{x}$ cannot be composed, since the target space of $\vec{y} = A\vec{x}$ is different from the domain of $\vec{z} = B\vec{y}$. (See Figure 6.) To put it more plainly: The output of $\vec{y} = A\vec{x}$ is not an acceptable input for the transformation $\vec{z} = B\vec{y}$. In this case, the matrix product BA is undefined.

Definition 2.3.1 Matrix multiplication

a. Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The product BA is defined if (and only if) $p = q$.

b. If B is an $n \times p$ matrix and A a $p \times m$ matrix, then the product BA is defined as the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$. This means that $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$, for all $\vec{x}$ in the vector space $\mathbb{R}^m$. The product BA is an $n \times m$ matrix.

Although this definition of matrix multiplication does not give us concrete instructions for computing the product of two numerically given matrices, such instructions can be derived easily from the definition.

As in Definition 2.3.1, let B be an $n \times p$ matrix and A a $p \times m$ matrix. Let's think about the columns of the matrix BA:

$$\begin{aligned}(i\text{th column of } BA) &= (BA)\vec{e}_i \\ &= B(A\vec{e}_i) \\ &= B(i\text{th column of } A).\end{aligned}$$

If we denote the columns of A by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, we can write

$$BA = B\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}.$$

Theorem 2.3.2 The columns of the matrix product

Let B be an $n \times p$ matrix and A a $p \times m$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$. Then, the product BA is

$$BA = B\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}.$$

To find BA, we can multiply B with the columns of A and combine the resulting vectors. ■

This is exactly how we computed the product

$$BA = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

on page 70, using approach (b).

For practice, let us multiply the same matrices in the reverse order. The first column of AB is $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 58 \end{bmatrix}$; the second is $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 25 \\ 66 \end{bmatrix}$. Thus,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 22 & 25 \\ 58 & 66 \end{bmatrix}.$$

Compare the two previous displays to see that $AB \neq BA$: Matrix multiplication is *noncommutative*. This should come as no surprise, in view of the fact that the matrix product represents a composite of transformations. Even for functions of one variable, the order in which we compose matters. Refer to the first example in this section and note that the functions $\cos(\sin(x))$ and $\sin(\cos(x))$ are different.

Theorem 2.3.3 **Matrix multiplication is noncommutative**

$AB \neq BA$, in general. However, at times it does happen that $AB = BA$; then we say that the matrices A and B *commute*. ∎

It is useful to have a formula for the ijth entry of the product BA of an $n \times p$ matrix B and a $p \times m$ matrix A.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be the columns of A. Then, by Theorem 2.3.2,

$$BA = B\begin{bmatrix} | & | & & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_j & \cdots & \vec{v}_m \\ | & | & & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_j & \cdots & B\vec{v}_m \\ | & | & & | & & | \end{bmatrix}.$$

The ijth entry of the product BA is the ith component of the vector $B\vec{v}_j$, which is the dot product of the ith row of B and $\vec{v}_j$, by Definition 1.3.7.

Theorem 2.3.4 **The entries of the matrix product**

Let B be an $n \times p$ matrix and A a $p \times m$ matrix. The ijth entry of BA is the dot product of the ith row of B with the jth column of A.

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pm} \end{bmatrix}$$

is the $n \times m$ matrix whose ijth entry is

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ip}a_{pj} = \sum_{k=1}^{p} b_{ik}a_{kj}.$$ ∎

EXAMPLE 1

$$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 6\cdot 1 + 7\cdot 3 & 6\cdot 2 + 7\cdot 5 \\ 8\cdot 1 + 9\cdot 3 & 8\cdot 2 + 9\cdot 5 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

We have done these computations before. (Where?) ∎

EXAMPLE 2 Compute the products BA and AB for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Interpret your answers geometrically, as composites of linear transformation. Draw composition diagrams.

Solution

$$BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that in this special example it turns out that $BA = -AB$.

From Section 2.2 we recall the following geometrical interpretations:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ represents the reflection about the vector } \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ represents the reflection about } \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ represents the rotation through } \frac{\pi}{2}; \text{ and}$$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ represents the rotation through } -\frac{\pi}{2}.$$

Let's use our standard L to show the effect of these transformations. See Figures 7 and 8. ■

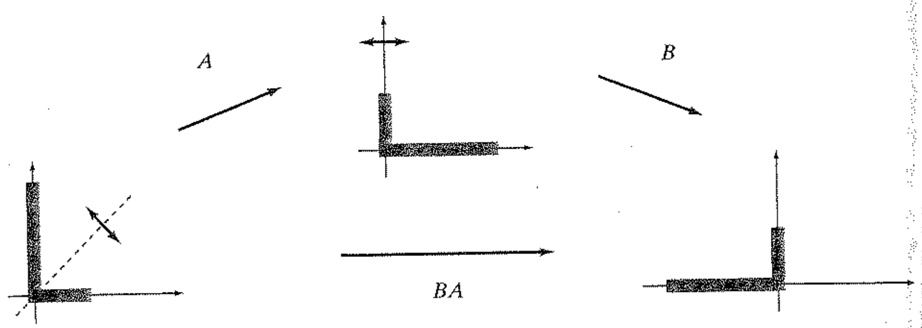


Figure 7

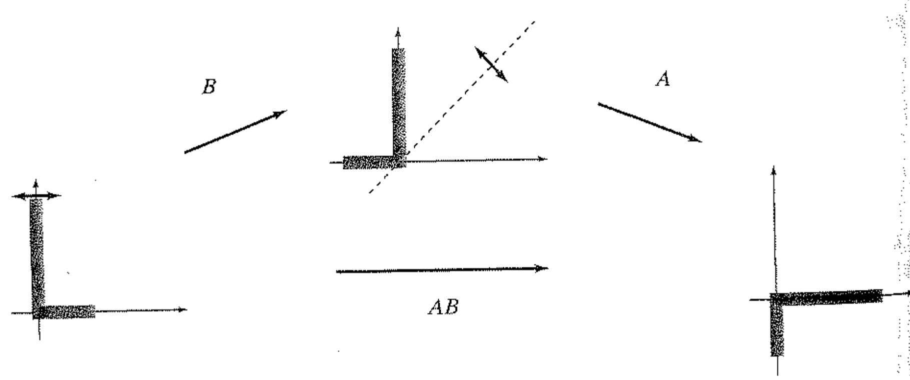


Figure 8

Matrix Algebra

Next let's discuss some algebraic rules for matrix multiplication.

- Composing a linear transformation with the identity transformation, on either side, leaves the transformation unchanged. (See Example 2.1.4.)

Theorem 2.3.5 Multiplying with the identity matrix

For an $n \times m$ matrix A,

$$AI_m = I_n A = A.$$

■

- If A is an $n \times p$ matrix, B a $p \times q$ matrix, and C a $q \times m$ matrix, what is the relationship between $(AB)C$ and $A(BC)$?

  One way to think about this problem (although perhaps not the most elegant one) is to write C in terms of its columns: $C = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$. Then

$$(AB)C = (AB)\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix} = \begin{bmatrix} (AB)\vec{v}_1 & (AB)\vec{v}_2 & \cdots & (AB)\vec{v}_m \end{bmatrix},$$

and

$$A(BC) = A\begin{bmatrix} B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \end{bmatrix} = \begin{bmatrix} A(B\vec{v}_1) & A(B\vec{v}_2) & \cdots & A(B\vec{v}_m) \end{bmatrix}.$$

Since $(AB)\vec{v}_i = A(B\vec{v}_i)$, by definition of the matrix product, we find that $(AB)C = A(BC)$.

Theorem 2.3.6 **Matrix multiplication is associative**

$$(AB)C = A(BC)$$

We can simply write ABC for the product $(AB)C = A(BC)$. ■

A more conceptual proof is based on the fact that the composition of functions is associative. The two linear transformations

$$T(\vec{x}) = \big((AB)C\big)\vec{x} \quad \text{and} \quad L(\vec{x}) = \big(A(BC)\big)\vec{x}$$

are identical because, by definition of matrix multiplication,

$$T(\vec{x}) = \big((AB)C\big)\vec{x} = (AB)(C\vec{x}) = A\big(B(C\vec{x})\big)$$

and

$$L(\vec{x}) = \big(A(BC)\big)\vec{x} = A\big((BC)\vec{x}\big) = A\big(B(C\vec{x})\big).$$

The domains and target spaces of the linear transformations defined by the matrices $A, B, C, BC, AB, A(BC)$, and $(AB)C$ are shown in Figure 9.

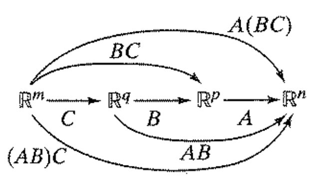


Figure 9

Theorem 2.3.7 **Distributive property for matrices**

If A and B are $n \times p$ matrices, and C and D are $p \times m$ matrices, then

$$A(C + D) = AC + AD, \quad \text{and}$$
$$(A + B)C = AC + BC.$$ ■

You will be asked to verify this property in Exercise 27.

Theorem 2.3.8 If A is an $n \times p$ matrix, B is a $p \times m$ matrix, and k is a scalar, then

$$(kA)B = A(kB) = k(AB).$$ ■

You will be asked to verify this property in Exercise 28.

Block Matrices (Optional)

In the popular puzzle Sudoku, one considers a 9×9 matrix A that is subdivided into nine 3×3 matrices called *blocks*. The puzzle setter provides some of the 81 entries of matrix A, and the objective is to fill in the remaining entries so that each row of A, each column of A, and each block contains each of the digits 1 through 9 exactly once.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

This is an example of a *block matrix* (or *partitioned matrix*), that is, a matrix that is partitioned into rectangular submatrices, called blocks, by means of horizontal and vertical lines that go all the way through the matrix.

The blocks need not be of equal size.

For example, we can partition the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \quad \text{as} \quad B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 6 & 7 & 9 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $B_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $B_{21} = \begin{bmatrix} 6 & 7 \end{bmatrix}$, and $B_{22} = [9]$.

A useful property of block matrices is the following:

Theorem 2.3.9 **Multiplying block matrices**

Block matrices can be multiplied as though the blocks were scalars (i.e., using the formula in Theorem 2.3.4):

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{np} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{p1} & B_{p2} & \cdots & B_{pj} & \cdots & B_{pm} \end{bmatrix}$$

is the block matrix whose ijth block is the matrix

$$A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \sum_{k=1}^{p} A_{ik}B_{kj},$$

provided that all the products $A_{ik}B_{kj}$ are defined. ■

Verifying this fact is left as an exercise. A numerical example follows.

EXAMPLE 3

$$\left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 7 & 8 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}[9] \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -3 & -3 & -3 \\ 8 & 10 & 12 \end{array}\right].$$

Compute this product without using a partition, and see whether or not you find the same result. ■

In this simple example, using blocks is somewhat pointless. Example 3 merely illustrates Theorem 2.3.9. In Example 2.4.7, we will see a more sensible application of the concept of block matrices.

EXERCISES 2.3

GOAL *Compute matrix products column by column and entry by entry. Interpret matrix multiplication in terms of the underlying linear transformations. Use the rules of matrix algebra. Multiply block matrices.*

If possible, compute the matrix products in Exercises 1 through 13, using paper and pencil.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

6. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -6 & 8 \\ 3 & -4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

14. For the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 5 \end{bmatrix},$$

determine which of the 25 matrix products AA, AB, $AC, \ldots, ED, EE$ are defined, and compute those that are defined.

Use the given partitions to compute the products in Exercises 15 and 16. Check your work by computing the same products without using a partition. Show all your work.

15. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 3 & 4 \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ 2 & 0 \\ \hline 3 & 4 \end{array}\right]$

16. $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 3 & 4 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right]$

In the Exercises 17 through 26, find all matrices that commute with the given matrix A.

17. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

19. $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

20. $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

21. $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

24. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

27. Prove the *distributive laws* for matrices:

$$A(C + D) = AC + AD$$

and

$$(A + B)C = AC + BC.$$

28. Consider an $n \times p$ matrix A, a $p \times m$ matrix B, and a scalar k. Show that

$$(kA)B = A(kB) = k(AB).$$

29. Consider the matrix

$$D_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

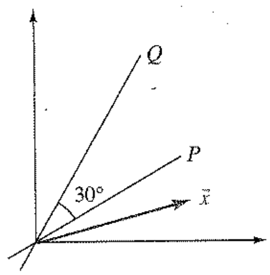
We know that the linear transformation $T(\vec{x}) = D_\alpha \vec{x}$ is a counterclockwise rotation through an angle α.

a. For two angles, α and β, consider the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Arguing geometrically, describe the linear transformations $\vec{y} = D_\alpha D_\beta \vec{x}$ and $\vec{y} = D_\beta D_\alpha \vec{x}$. Are the two transformations the same?

b. Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part (a)? Recall the trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

30. Consider the lines P and Q in $\mathbb{R}^2$ sketched below. Consider the linear transformation $T(\vec{x}) = \text{ref}_Q(\text{ref}_P(\vec{x}))$, that is, we first reflect $\vec{x}$ about P and then we reflect the result about Q.



a. For the vector $\vec{x}$ given in the figure, sketch $T(\vec{x})$. What angle do the vectors $\vec{x}$ and $T(\vec{x})$ enclose? What is the relationship between the lengths of $\vec{x}$ and $T(\vec{x})$?

b. Use your answer in part (a) to describe the transformation T geometrically, as a reflection, rotation, shear, or projection.

c. Find the matrix of T.

d. Give a geometrical interpretation of the linear transformation $L(\vec{x}) = \text{ref}_P(\text{ref}_Q(\vec{x}))$, and find the matrix of L.

31. Consider two matrices A and B whose product AB is defined. Describe the ith row of the product AB in terms of the rows of A and the matrix B.

32. Find all 2×2 matrices X such that $AX = XA$ for all 2×2 matrices A.

For the matrices A in Exercises 33 through 42, compute $A^2 = AA$, $A^3 = AAA$, and A^4. Describe the pattern that emerges, and use this pattern to find $A^{1,001}$. Interpret your answers geometrically, in terms of rotations, reflections, shears, and orthogonal projections.

33. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **35.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ **38.** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

39. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ **40.** $\frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

41. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **42.** $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

In Exercises 43 through 48, find a 2×2 matrix A with the given properties. (Hint: It helps to think of geometrical examples.)

43. $A \neq I_2$, $A^2 = I_2$ **44.** $A^2 \neq I_2$, $A^4 = I_2$

45. $A^2 \neq I_2$, $A^3 = I_2$

46. $A^2 = A$, all entries of A are nonzero.

47. $A^3 = A$, all entries of A are nonzero.

48. $A^{10} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

In Exercises 49 through 54, consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Compute the indicated products. Interpret these products geometrically, and draw composition diagrams, as in Example 2.

49. AF and FA **50.** CG and CG

51. FJ and JF **52.** JH and HJ

53. CD and DC **54.** BE and EB.

In Exercises 55 through 64, find all matrices X that satisfy the given matrix equation.

55. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

56. $X\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} X = I_2$

58. $X\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = I_2$ **59.** $X\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = I_2$

60. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = I_2$ **61.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} X = I_2$

62. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} X = I_2$ **63.** $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} X = I_3$

64. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} X = I_3$

65. Find all upper triangular 2×2 matrices X such that X^2 is the zero matrix.

66. Find all lower triangular 3×3 matrices X such that X^3 is the zero matrix.

67. Consider any 2×2 matrix A that represents a horizontal or vertical shear. Compute $(A - I_2)^2$. Explain your answer geometrically: If $\vec{x}$ is any vector in $\mathbb{R}^2$, what is $(A - I_2)^2\vec{x} = A^2\vec{x} - 2A\vec{x} + \vec{x}$?

68. Consider an $n \times m$ matrix A of rank n. Show that there exists an $m \times n$ matrix X such that $AX = I_n$. If $n < m$, how many such matrices X are there?

69. Consider an $n \times n$ matrix A of rank n. How many $n \times n$ matrices X are there such that $AX = I_n$?

2.4 The Inverse of a Linear Transformation

Let's first review the concept of an invertible function. As you read these abstract definitions, consider the examples in Figures 1 and 2, where X and Y are finite sets.

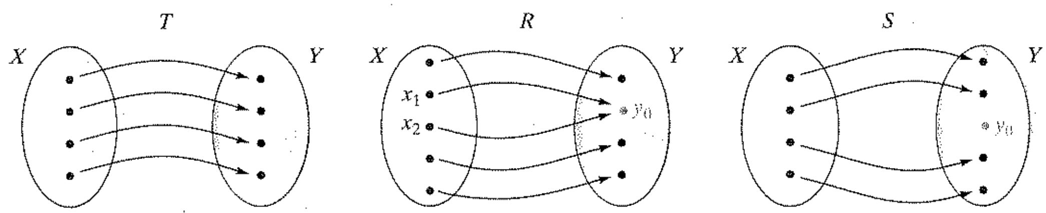


Figure 1 T is invertible. R is not invertible: The equation $R(x) = y_0$ has two solutions, x_1 and x_2. S is not invertible: There is no x such that $S(x) = y_0$.

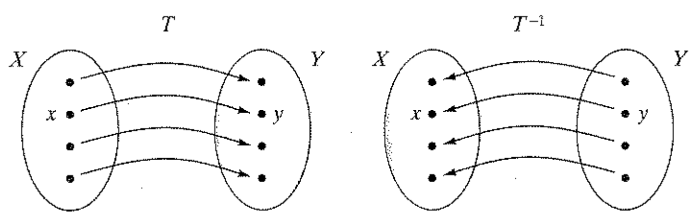


Figure 2 A function T and its inverse T^{-1}.

Definition 2.4.1 Invertible Functions

A function T from X to Y is called invertible if the equation $T(x) = y$ has a unique solution x in X for each y in Y.

In this case, the inverse T^{-1} from Y to X is defined by

$$T^{-1}(y) = \big(\text{the unique } x \text{ in } X \text{ such that } T(x) = y\big).$$

To put it differently, the equation

$$x = T^{-1}(y) \qquad \text{means that} \qquad y = T(x).$$

Note that

$$T^{-1}\big(T(x)\big) = x \qquad \text{and} \qquad T\big(T^{-1}(y)\big) = y$$

for all x in X and for all y in Y.

Conversely, if L is a function from Y to X such that

$$L\big(T(x)\big) = x \qquad \text{and} \qquad T\big(L(y)\big) = y$$

for all x in X and for all y in Y, then T is invertible and $T^{-1} = L$.

If a function T is invertible, then so is T^{-1} and $(T^{-1})^{-1} = T$.