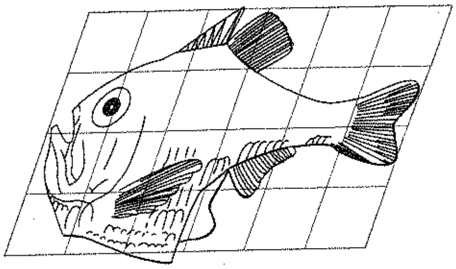


CHAPTER 2

Linear Transformations

2.1 Introduction to Linear Transformations and Their Inverses

Imagine yourself cruising in the Mediterranean as a crew member on a French coast guard boat, looking for evildoers. Periodically, your boat radios its position to headquarters in Marseille. You expect that communications will be intercepted. So, before you broadcast anything, you have to transform the actual position of the boat,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(x_1 for Eastern longitude, x_2 for Northern latitude), into an encoded position

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

You use the following code:

$$\begin{aligned} y_1 &= x_1 + 3x_2 \\ y_2 &= 2x_1 + 5x_2. \end{aligned}$$

For example, when the actual position of your boat is 5° E, 42° N, or

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 42 \end{bmatrix},$$

your encoded position will be

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 5 + 3 \cdot 42 \\ 2 \cdot 5 + 5 \cdot 42 \end{bmatrix} = \begin{bmatrix} 131 \\ 220 \end{bmatrix}.$$

(See Figure 1.)

The coding transformation can be represented as

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}},$$

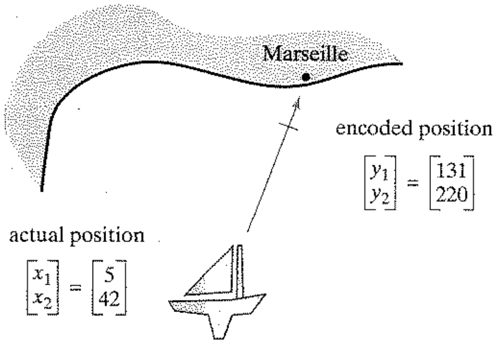


Figure 1

or, more succinctly, as

$$\vec{y} = A\vec{x}.$$

The matrix A is called the (*coefficient*) *matrix* of the transformation.

A transformation of the form

$$\vec{y} = A\vec{x}$$

is called a *linear transformation*. We will discuss this important concept in greater detail later in this section and throughout this chapter.

As the ship reaches a new position, the sailor on duty at headquarters in Marseille receives the encoded message

$$\vec{b} = \begin{bmatrix} 133 \\ 223 \end{bmatrix}.$$

He must determine the actual position of the boat. He will have to solve the linear system

$$A\vec{x} = \vec{b},$$

or, more explicitly,

$$\left| \begin{array}{r} x_1 + 3x_2 = 133 \\ 2x_1 + 5x_2 = 223 \end{array} \right|.$$

Here is his solution. Is it correct?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 43 \end{bmatrix}$$

As the boat travels on and dozens of positions are radioed in, the sailor gets a little tired of solving all those linear systems, and he thinks there must be a general formula to simplify the task. He wants to solve the system

$$\left| \begin{array}{r} x_1 + 3x_2 = y_1 \\ 2x_1 + 5x_2 = y_2 \end{array} \right|$$

when y_1 and y_2 are arbitrary constants, rather than particular numerical values. He is looking for the *decoding transformation*

$$\vec{y} \to \vec{x},$$

which is the *inverse*[1] of the coding transformation

$$\vec{x} \to \vec{y}.$$

The method of finding this solution is nothing new. We apply elimination as we have for a linear system with known values y_1 and y_2:

$$\left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ 2x_1 + 5x_2 = \quad y_2 \end{array} \right| \quad \begin{array}{c} \longrightarrow \\ -2\ (\mathrm{I}) \end{array} \quad \left| \begin{array}{l} x_1 + 3x_2 = \quad y_1 \\ \quad -x_2 = -2y_1 + y_2 \end{array} \right| \quad \begin{array}{c} \longrightarrow \\ \div(-1) \end{array}$$

$$\left| \begin{array}{l} x_1 + 3x_2 = y_1 \\ \quad x_2 = 2y_1 - y_2 \end{array} \right| \quad \begin{array}{c} -3\ (\mathrm{II}) \\ \longrightarrow \end{array} \quad \left| \begin{array}{l} x_1 \quad = -5y_1 + 3y_2 \\ \quad x_2 = \ \ 2y_1 - \ \ y_2 \end{array} \right|.$$

The formula for the decoding transformation is

$$\begin{aligned} x_1 &= -5y_1 + 3y_2, \\ x_2 &= \ \ 2y_1 - \ \ y_2, \end{aligned}$$

or

$$\vec{x} = B\vec{y}, \quad \text{where } B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Note that the decoding transformation is linear and that its coefficient matrix is

$$B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

The relationship between the two matrices A and B is shown in Figure 2.

$$\begin{tikzcd} \vec{x} \arrow[r, shift left, "\text{Coding, with matrix } A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}"] & \vec{y} \arrow[l, shift left, "\text{Decoding, with matrix } B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}"] \end{tikzcd}$$

Figure 2

Since the decoding transformation $\vec{x} = B\vec{y}$ is the inverse of the coding transformation $\vec{y} = A\vec{x}$, we say that the matrix B is the *inverse* of the matrix A. We can write this as $B = A^{-1}$.

Not all linear transformations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

are invertible. Suppose some ignorant officer chooses the code

$$\begin{aligned} y_1 &= \ \ x_1 + 2x_2 \\ y_2 &= 2x_1 + 4x_2 \end{aligned} \quad \text{with matrix} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

for the French coast guard boats. When the sailor in Marseille has to decode a position, for example,

$$\vec{b} = \begin{bmatrix} 89 \\ 178 \end{bmatrix},$$

---

[1] We will discuss the concept of the inverse of a transformation more systematically in Section 2.4.

he will be chagrined to discover that the system

$$\begin{vmatrix} x_1 + 2x_2 = 89 \\ 2x_1 + 4x_2 = 178 \end{vmatrix}$$

has infinitely many solutions, namely,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 89 - 2t \\ t \end{bmatrix},$$

where t is an arbitrary number.

Because this system does not have a unique solution, it is impossible to recover the actual position from the encoded position: The coding transformation and the coding matrix A are *noninvertible*. This code is useless!

Now let us discuss the important concept of *linear transformations* in greater detail. Since linear transformations are a special class of functions, it may be helpful to review the concept of a *function* first.

Consider two sets X and Y. A function T from X to Y is a rule that associates with each element x of X a unique element y of Y. The set X is called the *domain* of the function, and Y is its *target space*. We will sometimes refer to x as the *input* of the function and to y as its *output*. Figure 3 shows an example where domain X and target space Y are finite.

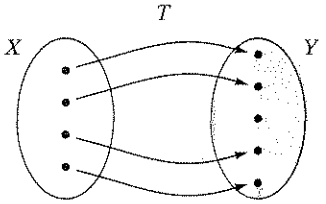


Figure 3 Domain X and target space Y of a function T.

In precalculus and calculus, you studied functions whose input and output are scalars (i.e., whose domain and target space are the real numbers $\mathbb{R}$ or subsets of $\mathbb{R}$); for example,

$$y = x^2, \quad f(x) = e^x, \quad g(t) = \frac{t^2 - 2}{t - 1}.$$

In multivariable calculus, you may have encountered functions whose input or output were vectors.

EXAMPLE 1

$$y = x_1^2 + x_2^2 + x_3^2$$

This formula defines a function from the vector space $\mathbb{R}^3$ to $\mathbb{R}$. The input is the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and the output is the scalar y. ■

EXAMPLE 2

$$\vec{r} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

This formula defines a function from $\mathbb{R}$ to the vector space $\mathbb{R}^3$, with input t and output $\vec{r}$. ■

We now return to the topic of linear transformations.

Definition 2.1.1 Linear transformations[2]

A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a *linear transformation* if there exists an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x},$$

for all $\vec{x}$ in the vector space $\mathbb{R}^m$.

It is important to note that a linear transformation is a special kind of *function*. The input and the output are both vectors. If we denote the output vector $T(\vec{x})$ by $\vec{y}$, we can write

$$\vec{y} = A\vec{x}.$$

Let us write this equation in terms of its components:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix},$$

or

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots &= \vdots \qquad \vdots \qquad \vdots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m. \end{aligned}$$

The output variables y_i are linear functions of the input variables x_j. In some branches of mathematics, a first-order function with a constant term, such as $y = 3x_1 - 7x_2 + 5x_3 + 8$, is called linear. Not so in linear algebra: The linear functions of m variables are those of the form $y = c_1x_1 + c_2x_2 + \cdots + c_mx_m$, for some coefficients $c_1, c_2, \ldots, c_m$. By contrast, a function such as $y = 3x_1 - 7x_2 + 5x_3 + 8$ is called *affine*.

EXAMPLE 3 The linear transformation

$$\begin{aligned} y_1 &= 7x_1 + 3x_2 - 9x_3 + 8x_4 \\ y_2 &= 6x_1 + 2x_2 - 8x_3 + 7x_4 \\ y_3 &= 8x_1 + 4x_2 \qquad\quad + 7x_4 \end{aligned}$$

(a function from $\mathbb{R}^4$ to $\mathbb{R}^3$) is represented by the 3×4 matrix

$$A = \begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}. \quad \blacksquare$$

---

[2] This is one of several possible definitions of a linear transformation; we could just as well have chosen the statement of Theorem 2.1.3 as the definition (as many texts do). This will be a recurring theme in this text: Most of the central concepts of linear algebra can be characterized in two or more ways. Each of these characterizations can serve as a possible definition; the other characterizations will then be stated as theorems, since we need to prove that they are equivalent to the chosen definition. Among these multiple characterizations, there is no "correct" definition (although mathematicians may have their favorite). Each characterization will be best suited for certain purposes and problems, while it is inadequate for others.

EXAMPLE 4 The coefficient matrix of the *identity transformation*

$$\begin{array}{lllll} y_1 = x_1 & & & \\ y_2 = & x_2 & & \\ \vdots & & \ddots & \\ y_n = & & & x_n \end{array}$$

(a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose output equals its input) is the $n \times n$ matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

All entries on the main diagonal are 1, and all other entries are 0. This matrix is called the *identity matrix* and is denoted by I_n:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and so on.}$$

■

We have already seen the identity matrix in other contexts. For example, we have shown that a linear system $A\vec{x} = \vec{b}$ of n equations with n unknowns has a unique solution if and only if $\text{rref}(A) = I_n$. (See Theorem 1.3.4.)

EXAMPLE 5 Consider the letter **L** (for Linear?) in Figure 4, made up of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Show the effect of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

on this letter, and describe the transformation in words.

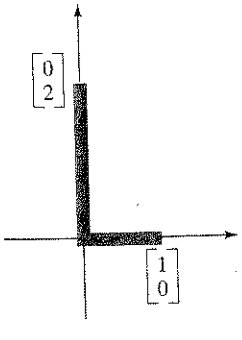


Figure 4

Solution

We have

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix},$$

as shown in Figure 5.

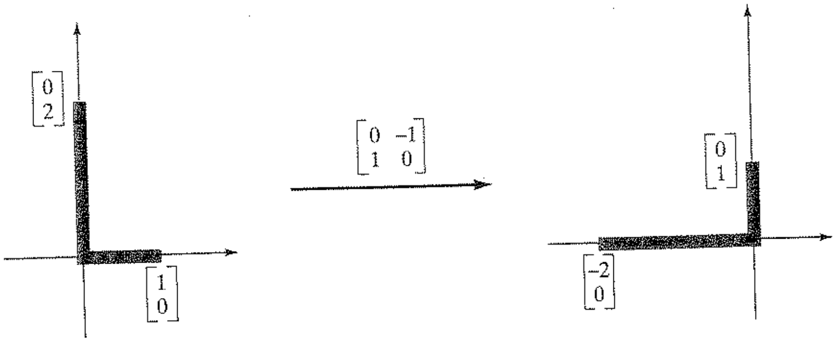


Figure 5

The **L** is rotated through an angle of 90° in the counterclockwise direction.

Let's examine the effect of transformation T on an arbitrary vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

We observe that the vectors $\vec{x}$ and $T(\vec{x})$ have the same length,

$$\sqrt{x_1^2 + x_2^2} = \sqrt{(-x_2)^2 + x_1^2},$$

and that they are perpendicular to one another, since the dot product equals zero (see Definition A.8 in the appendix):

$$\vec{x} \cdot T(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = -x_1x_2 + x_2x_1 = 0.$$

Paying attention to the signs of the components, we see that if $\vec{x}$ is in the first quadrant (meaning that x_1 and x_2 are both positive), then $T(\vec{x}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ is in the second quadrant. See Figure 6.

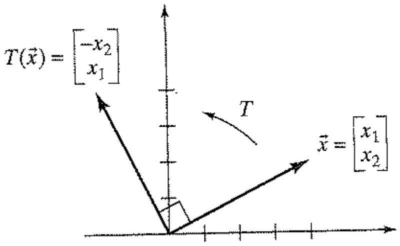


Figure 6

We can conclude that $T(\vec{x})$ is obtained by rotating vector $\vec{x}$ through an angle of 90° in the counterclockwise direction, as in the special cases $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ considered earlier. (Check that the rotation is indeed counterclockwise when $\vec{x}$ is in the second, third, or fourth quadrant.) ■

EXAMPLE 6 Consider the linear transformation $T(\vec{x}) = A\vec{x}$, with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where for simplicity we write $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ instead of $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

Solution

A straightforward computation shows that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\4\\7\end{bmatrix}$$

and

$$T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\6\\9\end{bmatrix}.$$

Note that $T\begin{bmatrix}1\\0\\0\end{bmatrix}$ is the first column of the matrix A and that $T\begin{bmatrix}0\\0\\1\end{bmatrix}$ is its third column. ■

We can generalize this observation:

Theorem 2.1.2 The columns of the matrix of a linear transformation

Consider a linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then, the matrix of T is

$$A = \begin{bmatrix}| & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & |\end{bmatrix}, \quad \text{where} \quad \vec{e}_i = \begin{bmatrix}0\\0\\ \vdots \\ 1 \\ \vdots \\ 0\end{bmatrix} \leftarrow i\text{th}.$$ ■

To justify this result, write

$$A = \begin{bmatrix}| & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & |\end{bmatrix}.$$

Then

$$T(\vec{e}_i) = A\vec{e}_i = \begin{bmatrix}| & | & & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_i & \cdots & \vec{v}_m \\ | & | & & | & & |\end{bmatrix}\begin{bmatrix}0\\0\\ \vdots \\ 1 \\ \vdots \\ 0\end{bmatrix} = \vec{v}_i,$$

by Theorem 1.3.8.

The vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ in the vector space $\mathbb{R}^m$ are sometimes referred to as the *standard vectors* in $\mathbb{R}^m$. The standard vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in $\mathbb{R}^3$ are often denoted by $\vec{i}, \vec{j}, \vec{k}$.

EXAMPLE 7 Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$.

a. What is the relationship between $T(\vec{v})$, $T(\vec{w})$, and $T(\vec{v}+\vec{w})$, where $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^m$?

b. What is the relationship between $T(\vec{v})$ and $T(k\vec{v})$, where $\vec{v}$ is a vector in $\mathbb{R}^m$ and k is a scalar?

Solution

a. Applying Theorem 1.3.10, we find that

$$T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w}).$$

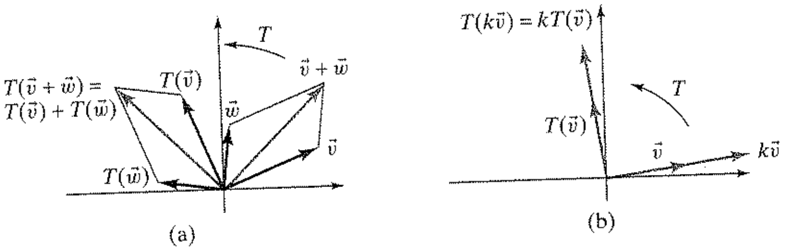
In words, the transform of the sum of two vectors equals the sum of the transforms.

b. Again, apply Theorem 1.3.10:

$$T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v}).$$

In words, the transform of a scalar multiple of a vector is the scalar multiple of the transform. ■

Figure 7 illustrates these two properties in the case of the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates a vector through an angle of 90° in the counterclockwise direction. (Compare this with Example 5.)



(a) (b)

Figure 7 (a) Illustrating the property $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$.
(b) Illustrating the property $T(k\vec{v}) = kT(\vec{v})$.

In Example 7, we saw that a linear transformation satisfies the two equations $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ and $T(k\vec{v}) = kT(\vec{v})$. Now we will show that the converse is true as well: Any transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ that satisfies these two equations is a linear transformation.

Theorem 2.1.3 Linear transformations

A transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is linear if (and only if)

a. $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$, for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^m$, and

b. $T(k\vec{v}) = kT(\vec{v})$, for all vectors $\vec{v}$ in $\mathbb{R}^m$ and all scalars k.

Proof In Example 7, we saw that a linear transformation satisfies the equations in (a) and (b). To prove the converse, consider a transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ that satisfies equations (a) and (b). We must show that there exists a matrix A such that

$T(\vec{x}) = A\vec{x}$, for all $\vec{x}$ in the vector space $\mathbb{R}^m$. Let $\vec{e}_1, \dots, \vec{e}_m$ be the standard vectors introduced in Theorem 2.1.2.

$$
\begin{aligned}
T(\vec{x}) = T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m) \\
&= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_m\vec{e}_m) \quad \text{(by property a)} \\
&= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m) \quad \text{(by property b)} \\
&= \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = A\vec{x}
\end{aligned}
$$

■

Here is an example illustrating Theorem 2.1.3.

EXAMPLE 8 Consider a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\vec{v}_1) = \frac{1}{2}\vec{v}_1$ and $T(\vec{v}_2) = 2\vec{v}_2$, for the vectors $\vec{v}_1$ and $\vec{v}_2$ sketched in Figure 8. On the same axes, sketch $T(\vec{x})$, for the given vector $\vec{x}$. Explain your solution.

Solution

Using a parallelogram, we can represent $\vec{x}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, as shown in Figure 9:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2.$$

By Theorem 2.1.3,

$$T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = \tfrac{1}{2}c_1\vec{v}_1 + 2c_2\vec{v}_2.$$

The vector $c_1\vec{v}_1$ is cut in half, and the vector $c_2\vec{v}_2$ is doubled, as shown in Figure 10.

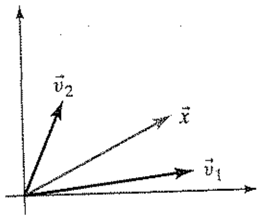


Figure 8

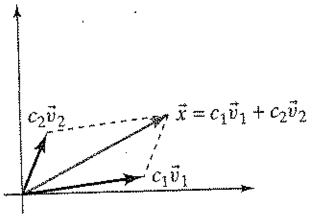


Figure 9

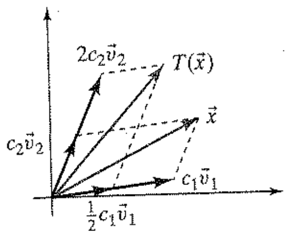


Figure 10

Imagine that vector $\vec{x}$ is drawn on a rubber sheet. Transformation T expands this sheet by a factor of 2 in the $\vec{v}_2$-direction and contracts it by a factor of 2 in the $\vec{v}_1$-direction. (We prefer "contracts by a factor of 2" to the awkward "expands by a factor of $\frac{1}{2}$.") ■

EXERCISES 2.1

GOAL *Use the concept of a linear transformation in terms of the formula $\vec{y} = A\vec{x}$, and interpret simple linear transformations geometrically. Find the inverse of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (if it exists). Find the matrix of a linear transformation column by column.*

Consider the transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined in Exercises 1 through 3. Which of these transformations are linear?

1. $y_1 = 2x_2$, $y_2 = x_2 + 2$, $y_3 = 2x_2$

2. $y_1 = 2x_2$, $y_2 = 3x_3$, $y_3 = x_1$

3. $y_1 = x_2 - x_3$, $y_2 = x_1x_3$, $y_3 = x_1 - x_2$

4. Find the matrix of the linear transformation

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3. \end{aligned}$$

5. Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}7\\11\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}6\\9\end{bmatrix},$$

$$\text{and} \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-13\\17\end{bmatrix}.$$

Find the matrix A of T.

6. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

7. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$T\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

8. Find the inverse of the linear transformation

$$\begin{aligned} y_1 &= x_1 + 7x_2 \\ y_2 &= 3x_1 + 20x_2. \end{aligned}$$

In Exercises 9 through 12, decide whether the given matrix is invertible. Find the inverse if it exists. In Exercise 12, the constant k is arbitrary.

9. $\begin{bmatrix}2 & 3\\6 & 9\end{bmatrix}$

10. $\begin{bmatrix}1 & 2\\4 & 9\end{bmatrix}$

11. $\begin{bmatrix}1 & 2\\3 & 9\end{bmatrix}$

12. $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$

✱ **13.** Prove the following facts:

a. The 2×2 matrix

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. (*Hint*: Consider the cases $a \neq 0$ and $a = 0$ separately.)

b. If

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}$$

is invertible, then

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}.$$

[The formula in part (b) is worth memorizing.]

14. **a.** For which values of the constant k is the matrix $\begin{bmatrix}2 & 3\\5 & k\end{bmatrix}$ invertible?

b. For which values of the constant k are all entries of $\begin{bmatrix}2 & 3\\5 & k\end{bmatrix}^{-1}$ integers?

(See Exercise 13.)

15. For which values of the constants a and b is the matrix

$$A = \begin{bmatrix}a & -b\\b & a\end{bmatrix}$$

invertible? What is the inverse in this case? (See Exercise 13.)

Give a geometric interpretation of the linear transformations defined by the matrices in Exercises 16 through 23. Show the effect of these transformations on the letter L considered in Example 5. In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically. See Exercise 13.

16. $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

17. $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

18. $\begin{bmatrix}0.5 & 0\\0 & 0.5\end{bmatrix}$

19. $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

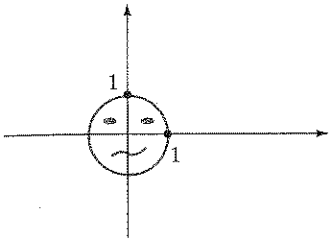
20. $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

21. $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

22. $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

23. $\begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}$

Consider the circular face in the accompanying figure. For each of the matrices A in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ on this face.

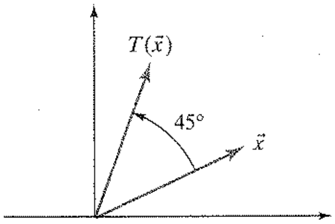


24. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
25. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
26. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
27. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
28. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
29. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
30. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

31. In Chapter 1, we mentioned that an old German bill shows the mirror image of Gauss's likeness. What linear transformation T can you apply to get the actual picture back?

32. Find an $n \times n$ matrix A such that $A\vec{x} = 3\vec{x}$, for all $\vec{x}$ in $\mathbb{R}^n$.

33. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector $\vec{x}$ through an angle of 45° in the counterclockwise direction, as shown in the following figure:



You are told that T is a linear transformation. (This will be shown in the next section.) Find the matrix of T.

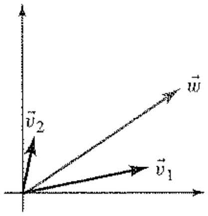
34. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector $\vec{x}$ through a given angle θ in the counterclockwise direction. (Compare this with Exercise 33.) You are told that T is linear. Find the matrix of T in terms of θ.

35. In the example about the French coast guard in this section, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

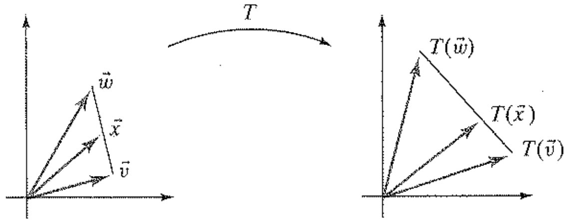
- When the actual position is $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$, they radio $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$.
- When the actual position is $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$, they radio $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?

36. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$ be three vectors in $\mathbb{R}^2$, as shown below. We are told that $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = 3\vec{v}_2$. On the same axes, sketch $T(\vec{w})$.



37. Consider a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that $\vec{v}$ and $\vec{w}$ are two arbitrary vectors in $\mathbb{R}^2$ and that $\vec{x}$ is a third vector whose endpoint is on the line segment connecting the endpoints of $\vec{v}$ and $\vec{w}$. Is the endpoint of the vector $T(\vec{x})$ necessarily on the line segment connecting the endpoints of $T(\vec{v})$ and $T(\vec{w})$? Justify your answer.

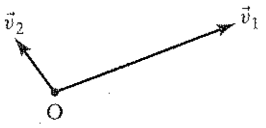


[*Hint*: We can write $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, for some scalar k between 0 and 1.]

We can summarize this exercise by saying that a linear transformation maps a line onto a line.

38. The two column vectors $\vec{v}_1$ and $\vec{v}_2$ of a 2×2 matrix A are shown in the accompanying sketch. Consider the linear transformation $T(\vec{x}) = A\vec{x}$, from $\mathbb{R}^2$ to $\mathbb{R}^2$. Sketch the vector

$$T\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$



39. Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_m T(\vec{e}_m),$$

where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ are the standard vectors in $\mathbb{R}^m$.

40. Describe all linear transformations from $\mathbb{R}$ $(= \mathbb{R}^1)$ to $\mathbb{R}$. What do their graphs look like?

41. Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$ $(= \mathbb{R}^1)$. What do their graphs look like?

42. When you represent a three-dimensional object graphically in the plane (on paper, the blackboard, or a computer screen), you have to transform spatial coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

into plane coordinates, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The simplest choice is a linear transformation, for example, the one given by the matrix

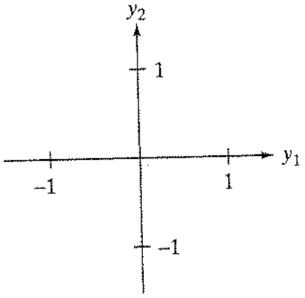
$$\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Include the images of the x_1, x_2, and x_3 axes in your sketch:



b. Represent the image of the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ in your figure in part (a). Explain.

c. Find all the points

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

that are transformed to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain.

43. **a.** Consider the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ (the dot product) from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T.

b. Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ linear? If so, find the matrix of T (in terms of the components of $\vec{v}$).

c. Conversely, consider a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$. Show that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{v} \cdot \vec{x}$, for all $\vec{x}$ in $\mathbb{R}^3$.

44. The cross product of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

(See Definition A.9 and Theorem A.11 in the Appendix.) Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\vec{v}$.

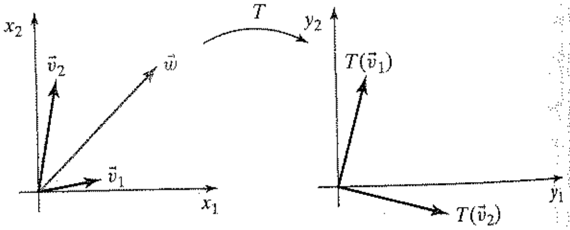
45. Consider two linear transformations $\vec{y} = T(\vec{x})$ and $\vec{z} = L(\vec{y})$, where T goes from $\mathbb{R}^m$ to $\mathbb{R}^p$ and L goes from $\mathbb{R}^p$ to $\mathbb{R}^n$. Is the transformation $\vec{z} = L\big(T(\vec{x})\big)$ linear as well? [The transformation $\vec{z} = L\big(T(\vec{x})\big)$ is called the *composite* of T and L.]

46. Let

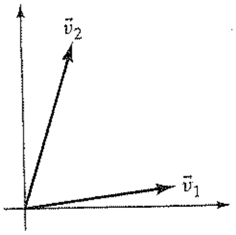
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Find the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$. (See Exercise 45.) [*Hint*: Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.]

47. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{w}$ in $\mathbb{R}^2$ and the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$ are shown in the accompanying figure. Sketch $T(\vec{w})$. Explain your answer.



48. Consider two linear transformations T and L from $\mathbb{R}^2$ to $\mathbb{R}^2$. We are told that $T(\vec{v}_1) = L(\vec{v}_1)$ and $T(\vec{v}_2) = L(\vec{v}_2)$ for the vectors $\vec{v}_1$ and $\vec{v}_2$ sketched below. Show that $T(\vec{x}) = L(\vec{x})$, for all vectors $\vec{x}$ in $\mathbb{R}^2$.



49. Some parking meters in downtown Geneva, Switzerland, accept 2 Franc and 5 Franc coins.

a. A parking officer collects 51 coins worth 144 Francs. How many coins are there of each kind?

b. Find the matrix A that transforms the vector

$$\begin{bmatrix} \text{number of 2 Franc coins} \\ \text{number of 5 Franc coins} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix}.$$

c. Is the matrix A in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

50. A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

a. King Hiero of Syracuse orders a crown from this goldsmith, with a total mass of 5 kilograms (or 5,000 grams), with the stipulation that the platinum alloy must make up at least 90% of the mass. The king's goldsmith delivers a beautiful piece, but the king's friend Archimedes has doubts about its purity. While taking a bath, he comes up with a method to check the composition of the crown (famously shouting "Eureka!" in the process, and running to the king's palace naked). Submerging the crown in water, he finds its volume to be 370 cubic centimeters. How much of each alloy went into this piece (by mass)? Is this goldsmith a crook?

b. Find the matrix A that transforms the vector

$$\begin{bmatrix} \text{mass of platinum alloy} \\ \text{mass of silver alloy} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix},$$

for any piece of jewelry this goldsmith makes.

c. Is the matrix A in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

51. The conversion formula $C = \frac{5}{9}(F - 32)$ from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.

a. Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ into the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$. (The second row of A will be $\begin{bmatrix} 0 & 1 \end{bmatrix}$.)

b. Is the matrix A in part (a) invertible? If so, find the inverse (use Exercise 13). Use the result to write a formula expressing F in terms of C.

52. In the financial pages of a newspaper, one can sometimes find a table (or matrix) listing the exchange rates between currencies. In this exercise we will consider a miniature version of such a table, involving only the Canadian dollar (C\$) and the South African Rand (ZAR). Consider the matrix

$$A = \begin{array}{cl} \begin{array}{cc} \text{C\$} & \text{ZAR} \end{array} & \\ \begin{bmatrix} 1 & 1/8 \\ 8 & 1 \end{bmatrix} & \begin{array}{l} \text{C\$} \\ \text{ZAR} \end{array} \end{array}$$

representing the fact that C\$1 is worth ZAR8 (as of June 2008).

a. After a trip you have C\$100 and ZAR1,600 in your pocket. We represent these two values in the vector $\vec{x} = \begin{bmatrix} 100 \\ 1{,}600 \end{bmatrix}$. Compute $A\vec{x}$. What is the practical significance of the two components of the vector $A\vec{x}$?

b. Verify that matrix A fails to be invertible. For which vectors $\vec{b}$ is the system $A\vec{x} = \vec{b}$ consistent? What is the practical significance of your answer? If the system $A\vec{x} = \vec{b}$ is consistent, how many solutions $\vec{x}$ are there? Again, what is the practical significance of the answer?

53. Consider a larger currency exchange matrix (see Exercise 52), involving four of the world's leading currencies: Euro (€), U.S. dollar (\$), Japanese yen (¥), and British pound (£).

$$A = \begin{array}{cl} \begin{array}{cccc} € & \$ & ¥ & £ \end{array} & \\ \begin{bmatrix} * & \frac{5}{8} & * & * \\ * & * & * & 2 \\ 170 & * & * & * \\ * & * & * & * \end{bmatrix} & \begin{array}{l} € \\ \$ \\ ¥ \\ £ \end{array} \end{array}$$

The entry a_{ij} gives the value of one unit of the jth currency, expressed in terms of the ith currency. For example, $a_{31} = 170$ means that €1 = ¥170 (as of June 2008). Find the exact values of the 13 missing entries of A (expressed as fractions).

54. Consider an arbitrary currency exchange matrix A (see Exercises 52 and 53).

a. What are the diagonal entries a_{ii} of A?

b. What is the relationship between a_{ij} and a_{ji}?

c. What is the relationship between a_{ik}, a_{kj}, and a_{ij}?

d. What is the rank of A? What is the relationship between A and $\text{rref}(A)$?