

1.3 On the Solutions of Linear Systems; Matrix Algebra

In this final section of Chapter 1, we will discuss two rather unrelated topics:

- First, we will examine how many solutions a system of linear equations can possibly have.
- Then, we will present some definitions and rules of matrix algebra.

The Number of Solutions of a Linear System

EXAMPLE 1 The reduced row-echelon forms of the augmented matrices of three systems are given. How many solutions are there in each case?

$$\text{a. } \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{b. } \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{c. } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Solution

- a. The third row represents the equation $0 = 1$, so that there are no solutions. We say that this system is *inconsistent*.
- b. The given augmented matrix represents the system

$$\begin{cases} x_1 + 2x_2 = 1 \\ x_3 = 2 \end{cases}, \quad \text{or,} \quad \begin{cases} x_1 = 1 - 2x_2 \\ x_3 = 2 \end{cases}.$$

We can assign an arbitrary value, t , to the free variable x_2 , so that the system has infinitely many solutions,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 2t \\ t \\ 2 \end{bmatrix}, \quad \text{where } t \text{ is an arbitrary constant.}$$

- c. Here there are no free variables, so that we have only one solution, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. ■

We can generalize our findings:¹⁶

Theorem 1.3.1 Number of solutions of a linear system

A system of equations is said to be *consistent* if there is at least one solution; it is *inconsistent* if there are no solutions.

A linear system is inconsistent if (and only if) the reduced row-echelon form of its augmented matrix contains the row $[0 \ 0 \ \dots \ 0 \ | \ 1]$, representing the equation $0 = 1$.

If a linear system is consistent, then it has either

- *infinitely many solutions* (if there is at least one free variable), or
- *exactly one solution* (if all the variables are leading). ■

¹⁶Starting in this section, we will number the definitions we give and the theorems we derive. The n th theorem stated in Section $p.q$ is labeled as Theorem $p.q.n$.

Example 1 illustrates what the number of leading 1's in the echelon form tells us about the number of solutions of a linear system. This observation motivates the following definition:

Definition 1.3.2 The rank of a matrix¹⁷

The rank of a matrix A is the number of leading 1's in $\text{rref}(A)$.

EXAMPLE 2 $\text{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 2$, since $\text{rref} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ■

Note that we have defined the rank of a *matrix* rather than the rank of a linear system. When relating the concept of rank to a linear system, we must be careful to specify whether we consider the coefficient matrix or the augmented matrix of the system.

EXAMPLE 3 Consider a system of n linear equations in m variables; its coefficient matrix A has the size $n \times m$. Show that

- The inequalities $\text{rank}(A) \leq n$ and $\text{rank}(A) \leq m$ hold.
- If $\text{rank}(A) = n$, then the system is consistent.
- If $\text{rank}(A) = m$, then the system has at most one solution.
- If $\text{rank}(A) < m$, then the system has either infinitely many solutions, or none.

To get a sense for the significance of these claims, take another look at Example 1.

Solution

- By definition of the reduced row-echelon form, there is at most one leading 1 in each of the n rows and in each of the m columns of $\text{rref}(A)$.
- If $\text{rank}(A) = n$, then there is a leading 1 in each row of $\text{rref}(A)$. This implies that the echelon form of the augmented matrix cannot contain the row $[0 \ 0 \ \dots \ 0 \ | \ 1]$. This system is consistent.
- For parts c and d, it is worth noting that

$$\left(\begin{array}{c} \text{number of} \\ \text{free variables} \end{array} \right) = \left(\begin{array}{c} \text{total number} \\ \text{of variables} \end{array} \right) - \left(\begin{array}{c} \text{number of} \\ \text{leading variables} \end{array} \right) = m - \text{rank}(A).$$

If $\text{rank}(A) = m$, then there are no free variables. Either the system is inconsistent or it has a unique solution (by Theorem 1.3.1).

- If $\text{rank}(A) < m$, then there are free variables ($m - \text{rank } A$ of them, to be precise). Therefore, either the system is inconsistent or it has infinitely many solutions (by Theorem 1.3.1). ■

EXAMPLE 4 Consider a linear system with fewer equations than variables. How many solutions could this system have?

Solution

Suppose there are n equations and m variables; we are told that $n < m$. Let A be the coefficient matrix of the system, of size $n \times m$. By Example 3a, we have

¹⁷This is a preliminary, rather technical definition. In Chapter 3, we will gain a better conceptual understanding of the rank.

$\text{rank}(A) \leq n < m$, so that $\text{rank}(A) < m$. There are free variables ($m - \text{rank } A$ of them), so that the system will have infinitely many solutions or no solutions at all. ■

Theorem 1.3.3 Systems with fewer equations than variables

A linear system with fewer equations than unknowns has either no solutions or infinitely many solutions.

To put it differently, if a linear system has a unique solution, then there must be at least as many equations as there are unknowns. ■

To illustrate this fact, consider two linear equations in three variables, with each equation defining a plane. Two different planes in space either intersect in a line or are parallel (see Figure 1), but they will never intersect at a point! This means that a system of two linear equations with three unknowns cannot have a unique solution.

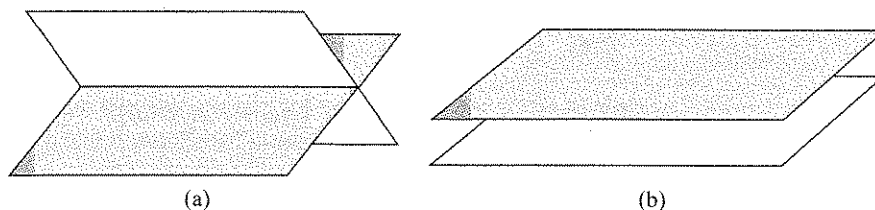


Figure 1 (a) Two planes intersect in a line. (b) Two parallel planes.

EXAMPLE 5 Consider a linear system of n equations in n variables. When does this system have a unique solution? Give your answer in terms of the rank of the coefficient matrix A .

Solution

If $\text{rank}(A) < n$, then there will be free variables ($n - \text{rank } A$ of them), so that the system has either no solutions or infinitely many solutions (see Example 3d). If $\text{rank}(A) = n$, then there are no free variables, so that there cannot be infinitely many solutions (see Example 3c). Furthermore, there must be a leading 1 in every row of $\text{rref}(A)$, so that the system is consistent (see Example 3b). We conclude that the system must have a unique solution in this case. ■

Theorem 1.3.4 Systems of n equations in n variables

A linear system of n equations in n variables has a unique solution if (and only if) the rank of its coefficient matrix A is n . In this case,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

the $n \times n$ matrix with 1's along the diagonal and 0's everywhere else. ■

Matrix Algebra

We will now introduce some basic definitions and rules of matrix algebra. Our presentation will be somewhat lacking in motivation at first, but it will be good to have these tools available when we need them in Chapter 2.

Sums and scalar multiples of matrices are defined entry by entry, as for vectors (see Definition A.1 in the Appendix).

Definition 1.3.5 Sums of matrices

The sum of two matrices of the same size is defined entry by entry:

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{nm} + b_{nm} \end{bmatrix}.$$

Scalar Multiples of Matrices

The product of a scalar with a matrix is defined entry by entry:

$$k \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}.$$

EXAMPLE 6 $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 4 \\ 9 & 8 & 5 \end{bmatrix}$ ■

EXAMPLE 7 $3 \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$ ■

The definition of the product of matrices is less straightforward; we will give the general definition only in Section 2.3.

Because vectors are special matrices (with only one row or only one column), it makes sense to start with a discussion of products of vectors. The reader may be familiar with the dot product of vectors.

Definition 1.3.6 Dot product of vectors

Consider two vectors \vec{v} and \vec{w} with components v_1, \dots, v_n and w_1, \dots, w_n , respectively. Here \vec{v} and \vec{w} may be column or row vectors, and they need not be of the same type. The dot product of \vec{v} and \vec{w} is defined to be the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n.$$

Note that our definition of the dot product isn't row-column-sensitive. The dot product does not distinguish between row and column vectors.

EXAMPLE 8 $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 = 11$ ■

Now we are ready to define the product $A\vec{x}$, where A is a matrix and \vec{x} is a vector, in terms of the dot product.

Definition 1.3.7 The product $A\vec{x}$

If A is an $n \times m$ matrix with row vectors $\vec{w}_1, \dots, \vec{w}_n$, and \vec{x} is a vector in \mathbb{R}^m , then

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}.$$

In words, the i th component of $A\vec{x}$ is the dot product of the i th row of A with \vec{x} .

Note that $A\vec{x}$ is a column vector with n components, that is, a vector in \mathbb{R}^n .

EXAMPLE 9 $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 + (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}.$ ■

EXAMPLE 10 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in \mathbb{R}^3 . ■

Note that the product $A\vec{x}$ is defined only if the number of *columns* of matrix A matches the number of components of vector \vec{x} :

$$\underbrace{\begin{matrix} n \times m & m \times 1 \\ A & \vec{x} \end{matrix}}_{n \times 1}.$$

EXAMPLE 11 The product $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ is undefined, because the number of columns of matrix A fails to match the number of components of vector \vec{x} . ■

In Definition 1.3.7, we express the product $A\vec{x}$ in terms of the *rows* of the matrix A . Alternatively, the product can be expressed in terms of the *columns*.

Let's take another look at Example 9:

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 + (-1) \cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 \\ 1 \cdot 3 \end{bmatrix} + \begin{bmatrix} 2 \cdot 1 \\ 0 \cdot 1 \end{bmatrix} + \begin{bmatrix} 3 \cdot 2 \\ (-1) \cdot 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{aligned}$$

We recognize that the expression $3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ involves the vectors

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, the columns of A , and the scalars $x_1 = 3$, $x_2 = 1$, $x_3 = 2$, the components of \vec{x} . Thus we can write

$$A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3.$$

We can generalize:

Theorem 1.3.8 The product $A\vec{x}$ in terms of the columns of A

If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \dots, \vec{v}_m$ and \vec{x} is a vector in \mathbb{R}^m with components x_1, \dots, x_m , then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.$$

Proof As usual, we denote the rows of A by $\vec{w}_1, \dots, \vec{w}_n$ and the entries by a_{ij} . It suffices to show that the i th component of $A\vec{x}$ is equal to the i th component of $x_1\vec{v}_1 + \dots + x_m\vec{v}_m$, for $i = 1, \dots, n$. Now

$$\begin{aligned} (\textit{i}th \textit{component of } A\vec{x}) &\stackrel{\text{Step 1}}{=} \vec{w}_i \cdot \vec{x} = a_{i1}x_1 + \dots + a_{im}x_m \\ &= x_1(\textit{i}th \textit{component of } \vec{v}_1) + \dots + x_m(\textit{i}th \textit{component of } \vec{v}_m) \\ &\stackrel{\text{Step 4}}{=} \textit{i}th \textit{component of } x_1\vec{v}_1 + \dots + x_m\vec{v}_m \end{aligned}$$

In Step 1 we are using Definition 1.3.7, and in Step 4 we are using the fact that vector addition and scalar multiplication are defined component by component. ■

EXAMPLE 12

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + (-4) \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix} - \begin{bmatrix} 8 \\ 20 \\ 32 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Note that something remarkable is happening here: Although A isn't the zero matrix and \vec{x} isn't the zero vector, the product $A\vec{x}$ is the zero vector. (By contrast, the product of any two nonzero *scalars* is nonzero.) ■

The formula for the product $A\vec{x}$ in Theorem 1.3.8 involves the expression $x_1\vec{v}_1 + \dots + x_m\vec{v}_m$, where $\vec{v}_1, \dots, \vec{v}_m$ are vectors in \mathbb{R}^n , and x_1, \dots, x_m are scalars. Such expressions come up very frequently in linear algebra; they deserve a name.

Definition 1.3.9 Linear combinations

A vector \vec{b} in \mathbb{R}^n is called a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_m$ in \mathbb{R}^n if there exist scalars x_1, \dots, x_m such that

$$\vec{b} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.$$

Note that the product $A\vec{x}$ is the linear combination of the columns of A with the components of \vec{x} as the coefficients:

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.$$

Take a good look at this equation, because it is the most frequently used formula in this text. Particularly in theoretical work, it will often be useful to write the product

$A\vec{x}$ as the linear combination $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$. Conversely, when dealing with a linear combination $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$, it will often be helpful to introduce the matrix

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \quad \text{and the vector} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

and then write $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = A\vec{x}$.

Next we present two rules concerning the product $A\vec{x}$. In Chapter 2 we will see that these rules play a central role in linear algebra.

Theorem 1.3.10 Algebraic rules for $A\vec{x}$

If A is an $n \times m$ matrix; \vec{x} and \vec{y} are vectors in \mathbb{R}^m ; and k is a scalar, then

- a. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$, and
- b. $A(k\vec{x}) = k(A\vec{x})$.

We will prove the first equation, leaving the second as Exercise 45.

Denote the i th row of A by \vec{w}_i . Then

$$\begin{aligned} (\textit{i} \text{th component of } A(\vec{x} + \vec{y})) &= \vec{w}_i \cdot (\vec{x} + \vec{y}) \stackrel{\text{Step 2}}{=} \vec{w}_i \cdot \vec{x} + \vec{w}_i \cdot \vec{y} \\ &= (\textit{i} \text{th component of } A\vec{x}) + (\textit{i} \text{th component of } A\vec{y}) \\ &= (\textit{i} \text{th component of } A\vec{x} + A\vec{y}). \end{aligned}$$

In Step 2 we are using a rule for dot products stated in Theorem A.5b, in the Appendix. ■

Our new tools of matrix algebra allow us to see linear systems in a new light, as illustrated in the next example. The definition of the product $A\vec{x}$ and the concept of a linear combination will be particularly helpful.

EXAMPLE 13 Consider the linear system

$$\begin{cases} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{cases}, \quad \text{with augmented matrix} \quad \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array} \right].$$

We can interpret the solution of this system as the intersection of two lines in the x_1x_2 -plane, as illustrated in Figure 2.

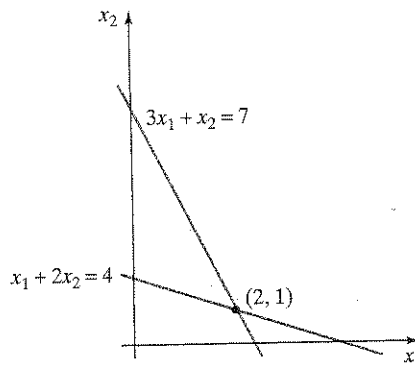


Figure 2

Alternatively, we can write the system in vector form, as

$$\begin{bmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{or} \quad x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

We see that solving this system amounts to writing the vector $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ as a *linear combination* of the vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (see Definition 1.3.9). The vector equation

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

and its solution can be represented geometrically, as shown in Figure 3. The problem amounts to resolving the vector $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ into two vectors parallel to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively, by means of a parallelogram.

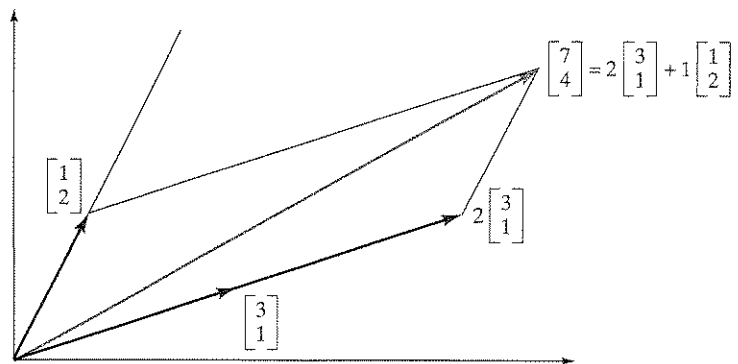


Figure 3

We can go further and write the linear combination

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{as} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so that the linear system

$$x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad \text{takes the form} \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix},$$

the *matrix form* of the linear system.

Note that we started out with the augmented matrix

$$\left[A \mid \vec{b} \right] = \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array} \right],$$

and we ended up writing the system as

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 4 \end{bmatrix}}_{\vec{b}}, \quad \text{or,} \quad A\vec{x} = \vec{b}.$$

We can generalize:

Theorem 1.3.11 Matrix form of a linear system

We can write the linear system with augmented matrix $[A \mid \vec{b}]$ in matrix form as

$$A\vec{x} = \vec{b}.$$

Note that the i th component of $A\vec{x}$ is $a_{i1}x_1 + \cdots + a_{im}x_m$, by Definition 1.3.7. Thus, the i th component of the equation $A\vec{x} = \vec{b}$ is

$$a_{i1}x_1 + \cdots + a_{im}x_m = b_i;$$

this is the i th equation of the system with augmented matrix $[A \mid \vec{b}]$.

EXAMPLE 14 Write the system

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 = 7 \\ 9x_1 + 4x_2 - 6x_3 = 8 \end{cases}$$

in matrix form.

Solution

The coefficient matrix is $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. The matrix form is

$$A\vec{x} = \vec{b}, \quad \text{or,} \quad \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

Now that we can write a linear system as a *single equation*, $A\vec{x} = \vec{b}$, rather than a list of simultaneous equations, we can think about it in new ways.

For example, if we have an equation $ax = b$ of *numbers*, we can divide both sides by a to find the solution x :

$$x = \frac{b}{a} = a^{-1}b \quad (\text{if } a \neq 0).$$

It is natural to ask whether we can take an analogous approach in the case of the equation $A\vec{x} = \vec{b}$. Can we “divide by A ,” in some sense, and write

$$\vec{x} = \frac{\vec{b}}{A} = A^{-1}\vec{b}?$$

This issue of the invertibility of a matrix will be one of the main themes of Chapter 2.

EXERCISES 1.3

GOAL Use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system. Apply the definition of the rank of a matrix. Compute the product $A\vec{x}$ in terms of the rows or the columns of A . Represent a linear system in vector or in matrix form.

- The reduced row-echelon forms of the augmented matrices of three systems are given below. How many solutions does each system have?

a. $\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & | & 5 \\ 0 & 1 & | & 6 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$

Find the rank of the matrices in Exercises 2 through 4.

2. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

5. a. Write the system

$$\begin{cases} x + 2y = 7 \\ 3x + y = 11 \end{cases}$$

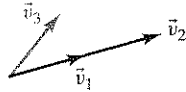
in vector form.

b. Use your answer in part (a) to represent the system geometrically. Solve the system and represent the solution geometrically.

6. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in \mathbb{R}^2 (sketched in the accompanying figure). Vectors \vec{v}_1 and \vec{v}_2 are parallel. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

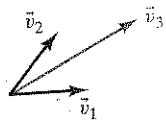
have? Argue geometrically.



7. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in \mathbb{R}^2 shown in the accompanying sketch. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

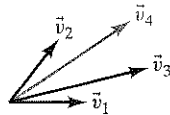
have? Argue geometrically.



8. Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in \mathbb{R}^2 shown in the accompanying sketch. Arguing geometrically, find two solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know that this system has in fact infinitely many solutions?



9. Write the system

$$\begin{cases} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 4 \\ 7x + 8y + 9z = 9 \end{cases}$$

in matrix form.

Compute the dot products in Exercises 10 through 12 (if the products are defined).

10. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 9 & 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 13 through 15 using paper and pencil. In each case, compute the product two ways: in terms of the columns of A (Theorem 1.3.8) and in terms of the rows of A (Definition 1.3.7).

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ 14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 16 through 19 using paper and pencil (if the products are defined).

16. $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ 17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 19. $\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

20. a. Find $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}$.

b. Find $9 \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

21. Use technology to compute the product

$$\begin{bmatrix} 1 & 7 & 8 & 9 \\ 1 & 2 & 9 & 1 \\ 1 & 5 & 1 & 5 \\ 1 & 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 5 \\ 6 \end{bmatrix}$$

22. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

24. Let A be a 4×4 matrix, and let \vec{b} and \vec{c} be two vectors in \mathbb{R}^4 . We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

25. Let A be a 4×4 matrix, and let \vec{b} and \vec{c} be two vectors in \mathbb{R}^4 . We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?
26. Let A be a 4×3 matrix, and let \vec{b} and \vec{c} be two vectors in \mathbb{R}^4 . We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?
27. If the rank of a 4×4 matrix A is 4, what is $\text{rref}(A)$?

28. If the rank of a 5×3 matrix A is 3, what is $\text{rref}(A)$?

In Problems 29 through 32, let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

29. Find a *diagonal* matrix A such that $A\vec{x} = \vec{y}$.
30. Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.
31. Find an *upper triangular* matrix A such that $A\vec{x} = \vec{y}$. Also, it is required that all the entries of A on and above the diagonal be nonzero.
32. Find a matrix A with all nonzero entries such that $A\vec{x} = \vec{y}$.
33. Let A be the $n \times n$ matrix with all 1's on the diagonal and all 0's above and below the diagonal. What is $A\vec{x}$, where \vec{x} is a vector in \mathbb{R}^n ?
34. We define the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in \mathbb{R}^3 .

a. For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$

compute $A\vec{e}_1$, $A\vec{e}_2$, and $A\vec{e}_3$.

b. If B is an $n \times 3$ matrix with columns \vec{v}_1 , \vec{v}_2 , and \vec{v}_3 , what is $B\vec{e}_1$, $B\vec{e}_2$, $B\vec{e}_3$?

35. In \mathbb{R}^m , we define

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component.}$$

If A is an $n \times m$ matrix, what is $A\vec{e}_i$?

36. Find a 3×3 matrix A such that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

$$\text{and } A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

37. Find all vectors \vec{x} such that $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

38. a. Using technology, generate a random 3×3 matrix A . (The entries may be either single-digit integers or numbers between 0 and 1, depending on the technology you are using.) Find $\text{rref}(A)$. Repeat this experiment a few times.
- b. What does the reduced row-echelon form of most 3×3 matrices look like? Explain.

39. Repeat Exercise 38 for 3×4 matrices.

40. Repeat Exercise 38 for 4×3 matrices.

41. How many solutions do most systems of three linear equations with three unknowns have? Explain in terms of your work in Exercise 38.

42. How many solutions do most systems of three linear equations with four unknowns have? Explain in terms of your work in Exercise 39.

43. How many solutions do most systems of four linear equations with three unknowns have? Explain in terms of your work in Exercise 40.

44. Consider an $n \times m$ matrix A with more rows than columns ($n > m$). Show that there is a vector \vec{b} in \mathbb{R}^n such that the system $A\vec{x} = \vec{b}$ is inconsistent.

45. Consider an $n \times m$ matrix A , a vector \vec{x} in \mathbb{R}^m , and a scalar k . Show that

$$A(k\vec{x}) = k(A\vec{x}).$$

46. Find the rank of the matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$

where a , d , and f are nonzero, and b , c , and e are arbitrary numbers.

47. A linear system of the form

$$A\vec{x} = \vec{0}$$

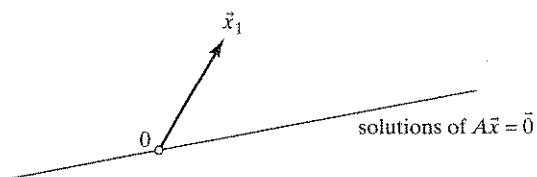
is called *homogeneous*. Justify the following facts:

a. All homogeneous systems are consistent.

- b. A homogeneous system with fewer equations than unknowns has infinitely many solutions.
- c. If \vec{x}_1 and \vec{x}_2 are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_2$ is a solution as well.
- d. If \vec{x} is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and k is an arbitrary constant, then $k\vec{x}$ is a solution as well.

48. Consider a solution \vec{x}_1 of the linear system $A\vec{x} = \vec{b}$. Justify the facts stated in parts (a) and (b):

- a. If \vec{x}_h is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.
- b. If \vec{x}_2 is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.
- c. Now suppose A is a 2×2 matrix. A solution vector \vec{x}_1 of the system $A\vec{x} = \vec{b}$ is shown in the accompanying figure. We are told that the solutions of the system $A\vec{x} = \vec{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $A\vec{x} = \vec{b}$.



If you are puzzled by the generality of this problem, think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

49. Consider the accompanying table. For some linear systems $A\vec{x} = \vec{b}$, you are given either the rank of the coefficient matrix A , or the rank of the augmented matrix $[A \mid \vec{b}]$. In each case, state whether the system could have no solution, one solution, or infinitely many solutions. There may be more than one possibility for some systems. Justify your answers.

	Number of Equations	Number of Unknowns	Rank of A	Rank of $[A \mid \vec{b}]$
a.	3	4	—	2
b.	4	3	3	—
c.	4	3	—	4
d.	3	4	3	—

50. Consider a linear system $A\vec{x} = \vec{b}$, where A is a 4×3 matrix. We are told that $\text{rank}[A \mid \vec{b}] = 4$. How many solutions does this system have?

51. Consider an $n \times m$ matrix A , an $r \times s$ matrix B , and a vector \vec{x} in \mathbb{R}^p . For which values of n, m, r, s , and p is the product

$$A(B\vec{x})$$

defined?

52. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Can you find a 2×2 matrix C such that

$$A(B\vec{x}) = C\vec{x},$$

for all vectors \vec{x} in \mathbb{R}^2 ?

53. If A and B are two $n \times m$ matrices, is

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

for all \vec{x} in \mathbb{R}^m ?

54. Consider two vectors \vec{v}_1 and \vec{v}_2 in \mathbb{R}^3 that are not parallel. Which vectors in \mathbb{R}^3 are linear combinations of \vec{v}_1 and \vec{v}_2 ? Describe the set of these vectors geometrically. Include a sketch in your answer.

55. Is the vector $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ a linear combination of

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}?$$

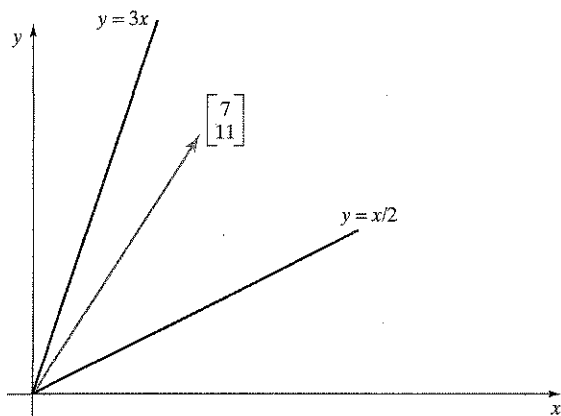
56. Is the vector

$$\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}?$$

57. Express the vector $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ as the sum of a vector on the line $y = 3x$ and a vector on the line $y = x/2$.



58. For which values of the constants b and c is the vector

$$\begin{bmatrix} 3 \\ b \\ c \end{bmatrix} \text{ a linear combination of } \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} ?$$

59. For which values of the constants c and d is $\begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix}$ a linear

$$\text{combination of } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} ?$$

60. For which values of the constants $a, b, c,$ and d is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

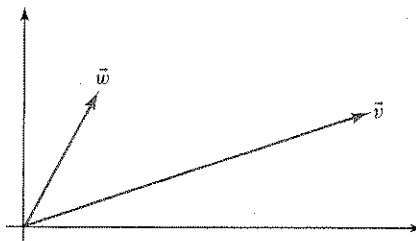
$$\text{a linear combination of } \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix} ?$$

61. For which values of the constant c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear

$$\text{combination of } \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} ?$$

62. For which values of the constant c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$, where a and b are arbitrary constants?

In Exercises 63 through 67, consider the vectors \vec{v} and \vec{w} in the accompanying figure.



63. Give a geometrical description of the set of all vectors of the form $\vec{v} + c\vec{w}$, where c is an arbitrary real number.
64. Give a geometrical description of the set of all vectors of the form $\vec{v} + c\vec{w}$, where $0 \leq c \leq 1$.
65. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.
66. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $a + b = 1$.
67. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $0 \leq a, 0 \leq b,$ and $a + b \leq 1$.
68. Give a geometrical description of the set of all vectors \vec{u} in \mathbb{R}^2 such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$.
69. Solve the linear system

$$\begin{cases} y + z = a \\ x + z = b \\ x + y = c \end{cases}$$

where $a, b,$ and c are arbitrary constants.

70. Let A be the $n \times n$ matrix with 0's on the main diagonal, and 1's everywhere else. For an arbitrary vector \vec{b} in \mathbb{R}^n , solve the linear system $A\vec{x} = \vec{b}$, expressing the components x_1, \dots, x_n of \vec{x} in terms of the components of \vec{b} . (See Exercise 69 for the case $n = 3$.)

TRUE OR FALSE?¹⁸

Determine whether the statements that follow are true or false, and justify your answer.

1. There exists a 3×4 matrix with rank 4.
2. If A is a 3×4 matrix and vector \vec{v} is in \mathbb{R}^4 , then vector $A\vec{v}$ is in \mathbb{R}^3 .
3. If the 4×4 matrix A has rank 4, then any linear system with coefficient matrix A will have a unique solution.
4. There exists a system of three linear equations with three unknowns that has exactly three solutions.
5. There exists a 5×5 matrix A of rank 4 such that the system $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.
6. If matrix A is in rref, then at least one of the entries in each column must be 1.
7. If A is an $n \times n$ matrix and \vec{x} is a vector in \mathbb{R}^n , then the product $A\vec{x}$ is a linear combination of the columns of matrix A .
8. If vector \vec{u} is a linear combination of vectors \vec{v} and \vec{w} , then we can write $\vec{u} = a\vec{v} + b\vec{w}$ for some scalars a and b .
9. Matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in rref.
10. A system of four linear equations in three unknowns is always inconsistent.
11. If A is a nonzero matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then the rank of A must be 2.
12. $\text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = 3$
13. The system $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent for all 4×3 matrices A .

¹⁸We will conclude each chapter (except for Chapter 9) with some true-false questions, over 400 in all. We will start with a group of about 10 straightforward statements that refer directly to definitions and theorems given in the chapter. Then there may be some computational exercises, and the remaining ones are more conceptual, calling for independent reasoning. In some chapters, a few of the problems toward the end can be quite challenging. Don't expect a balanced coverage of all the topics; some concepts are better suited for this kind of questioning than others.

14. There exists a 2×2 matrix A such that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

15. $\text{rank} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = 2$

$$16. \begin{bmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \\ 21 \end{bmatrix}$$

17. There exists a matrix A such that $A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

18. Vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of vectors

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

19. The system $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is inconsistent.

20. There exists a 2×2 matrix A such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

21. If A and B are any two 3×3 matrices of rank 2, then A can be transformed into B by means of elementary row operations.

22. If vector \vec{u} is a linear combination of vectors \vec{v} and \vec{w} , and \vec{v} is a linear combination of vectors \vec{p} , \vec{q} , and \vec{r} , then \vec{u} must be a linear combination of \vec{p} , \vec{q} , \vec{r} , and \vec{w} .

23. A linear system with fewer unknowns than equations must have infinitely many solutions or none.

24. The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.

25. If the system $A\vec{x} = \vec{b}$ has a unique solution, then A must be a square matrix.

26. If A is any 4×3 matrix, then there exists a vector \vec{b} in \mathbb{R}^4 such that the system $A\vec{x} = \vec{b}$ is inconsistent.

27. There exist scalars a and b such that matrix

$$\begin{bmatrix} 0 & 1 & a \\ -1 & 0 & b \\ -a & -b & 0 \end{bmatrix}$$

has rank 3.

28. If \vec{v} and \vec{w} are vectors in \mathbb{R}^4 , then \vec{v} must be a linear combination of \vec{v} and \vec{w} .
29. If \vec{u} , \vec{v} , and \vec{w} are nonzero vectors in \mathbb{R}^2 , then \vec{w} must be a linear combination of \vec{u} and \vec{v} .
30. If \vec{v} and \vec{w} are vectors in \mathbb{R}^4 , then the zero vector in \mathbb{R}^4 must be a linear combination of \vec{v} and \vec{w} .
31. There exists a 4×3 matrix A of rank 3 such that $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{0}$.
32. The system $A\vec{x} = \vec{b}$ is inconsistent if (and only if) $\text{rref}(A)$ contains a row of zeros.
33. If A is a 4×3 matrix of rank 3 and $A\vec{v} = A\vec{w}$ for two vectors \vec{v} and \vec{w} in \mathbb{R}^3 , then vectors \vec{v} and \vec{w} must be equal.
34. If A is a 4×4 matrix and the system $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ has a unique solution, then the system $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.
35. If vector \vec{u} is a linear combination of vectors \vec{v} and \vec{w} , then \vec{w} must be a linear combination of \vec{u} and \vec{v} .
36. If $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, then the equation $\vec{w} = 2\vec{u} + 3\vec{v}$ must hold.
37. If A and B are matrices of the same size, then the formula $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$ must hold.
38. If A and B are any two $n \times n$ matrices of rank n , then A can be transformed into B by means of elementary row operations.
39. If a vector \vec{v} in \mathbb{R}^4 is a linear combination of \vec{u} and \vec{w} , and if A is a 5×4 matrix, then $A\vec{v}$ must be a linear combination of $A\vec{u}$ and $A\vec{w}$.
40. If matrix E is in reduced row-echelon form, and if we omit a row of E , then the remaining matrix must be in reduced row-echelon form as well.
41. The linear system $A\vec{x} = \vec{b}$ is consistent if (and only if) $\text{rank}(A) = \text{rank} \begin{bmatrix} A & \vec{b} \end{bmatrix}$.
42. If A is a 3×4 matrix of rank 3, then the system $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ must have infinitely many solutions.
43. If two matrices A and B have the same reduced row-echelon form, then the equations $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ must have the same solutions.
44. If matrix E is in reduced row-echelon form, and if we omit a column of E , then the remaining matrix must be in reduced row-echelon form as well.
45. If A and B are two 2×2 matrices such that the equations $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions, then $\text{rref}(A)$ must be equal to $\text{rref}(B)$.