Math 151
Implicit Differentiation Workalong

Based on the Khan Academy videos

Video 1: "Implicit Differentiation" with URL https://youtu.be/sL6MC-IKOrw

Concept 1: (start 0:00 and end 2:00 Implicit vs. explicit equations

Example of an explicit equation: $y = x^2 + 2x + 3$

Example of an implicit equation: $x^2 + y^2 = 100$

$\Rightarrow y^2 = 100 - x^2$

$\Rightarrow y = \pm\sqrt{100 - x^2}$

Example 1: (start 2:01 and end 11:25 Find the derivative $\frac{dy}{dx}$ of the implicitly defined equation $x^2 + y^2 = 100$

$x^2 + y^2 = 100$

$\Rightarrow \frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[100]$

$\Rightarrow \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2]$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow 2y\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

or you can write this explicitly as

$\frac{dy}{dx} = \frac{-x}{\pm\sqrt{100 - x^2}}$

Aside on the chain rule

$g = y^2$

$\frac{d}{dx}[g] = \frac{d}{dx}[y^2]$

$\frac{dg}{dx} = \frac{d}{dx}[y^2]$

$= \frac{dg}{dy}\frac{dy}{dx}$

$= 2y\frac{dy}{dx}$

Example 2: (start 11:26 and end 15:56 Find the derivative $\frac{dy}{dx}$ of the implicitly defined equation $y = x^x$.

YOU MAY SKIP THIS EXAMPLE.

THIS REQUIRES LOGS WHICH WE DON'T INTRODUCE UNTIL LATER

Video 2: "Implicit Differentiation (part 2)" with URL https://youtu.be/PUsMyhds5S4

Example 3: (0:00 - 10:44) Find the derivative $\frac{dy}{dx}$ of the equation $y^3 - xy^2 + \cos(xy) = 2$.

$y^3 - xy^2 + \cos(xy) = 2$

$\Rightarrow \frac{d}{dx}\left[y^3 - xy^2 + \cos(xy)\right] = \frac{d}{dx}[2]$

$\Rightarrow 3y^2\frac{dy}{dx} - \left(1\cdot y^2 + 2y\frac{dy}{dx}x\right) + -\sin(xy)\cdot\frac{d}{dx}(xy) = 0$

$\Rightarrow 3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} - \sin(xy)\left(1\cdot y + x\frac{dy}{dx}\right) = 0$

$\Rightarrow 3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} - y\sin(xy) - x\frac{dy}{dx}\sin(xy) = 0$

$\Rightarrow \frac{dy}{dx}\left[3y^2 - 2xy - x\sin(xy)\right] = y^2 + y\sin(xy)$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 + y\sin(xy)}{3y^2 - 2xy - x\sin(xy)}$

Aside

$g = xy$

$\frac{d}{dg}\left[\cos(g)\right] = -\sin(g)$

Video 3: "More implicit differentiation" with URL https://youtu.be/hrg1hCzg3W0

Example 4: (0:00 - 6:45) Find the derivative $\frac{dy}{dx}$ of the equation $x^2 - 24xy + 16y^2 - 400x - 300y = 0$.

$x^2 - 24xy + 16y^2 - 400x - 300y = 0$

$\Rightarrow \frac{d}{dx}\left[x^2 - 24xy + 16y^2 - 400x - 300y\right] = \frac{d}{dx}[0]$

$\Rightarrow 2x - \underbrace{24(1y + xy')}_{-24y - 24xy'} + 32yy' - 400 - 300y' = 0$

$\Rightarrow y'(32y - 24x - 300) = 24y - 2x + 400$

$\Rightarrow y' = \frac{24y - 2x + 400}{32y - 24x - 300}$

$= \frac{12y - x + 200}{16y - 12x - 150}$

Aside

Note: $\frac{dy}{dx} = y'$

Example 5: (6:46 - 11:29) Find the derivative $\frac{dy}{dx}$ of the equation $(x-y)^2 = 8(y-6)$.

$$\Rightarrow \frac{d}{dx}\left[(x-y)^2\right] = \frac{d}{dx}\left[8(y-6)\right]$$

Note: $8(y-6) = 8y - 48$

$$\Rightarrow 2(x-y)\left(1 - \frac{dy}{dx}\right) = 8\frac{dy}{dx}$$

$$\Rightarrow (2x - 2y)\left(1 - \frac{dy}{dx}\right) = 8\frac{dy}{dx}$$

$$\Rightarrow 2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} = 8\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2y - 2x - 8) = 2y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y-2x}{2y-2x-8}$$

$$= \frac{y-x}{y-x-4}$$