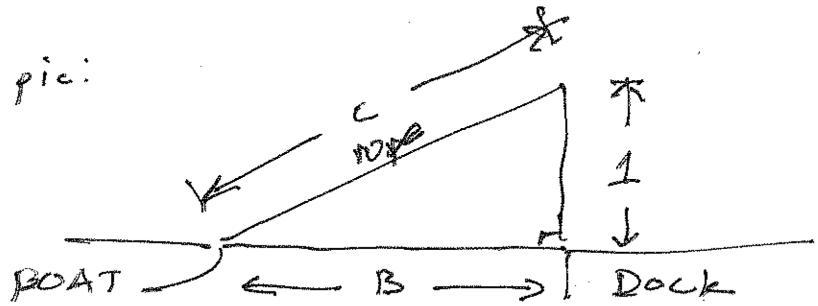
Related Rates

Example 1: A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

(1) Draw a pic:



(2) relate the variables: $1^2 + B^2 = c^2$ ← B & c are fcns of time.

Pythag. Thm.

(3) we know: $\frac{dc}{dt} = -1$ m/s

(4) we want: $\frac{dB}{dt}$

(5) use implicit differentiation

$\Rightarrow \frac{d}{dt}(1 + B^2) = \frac{d}{dt} c^2$

$\Rightarrow 2B \cdot \frac{dB}{dt} = 2c \frac{dc}{dt}$

$\Rightarrow B \cdot \frac{dB}{dt} = c \frac{dc}{dt}$ we know $\frac{dc}{dt} = -1$

$B = 8$

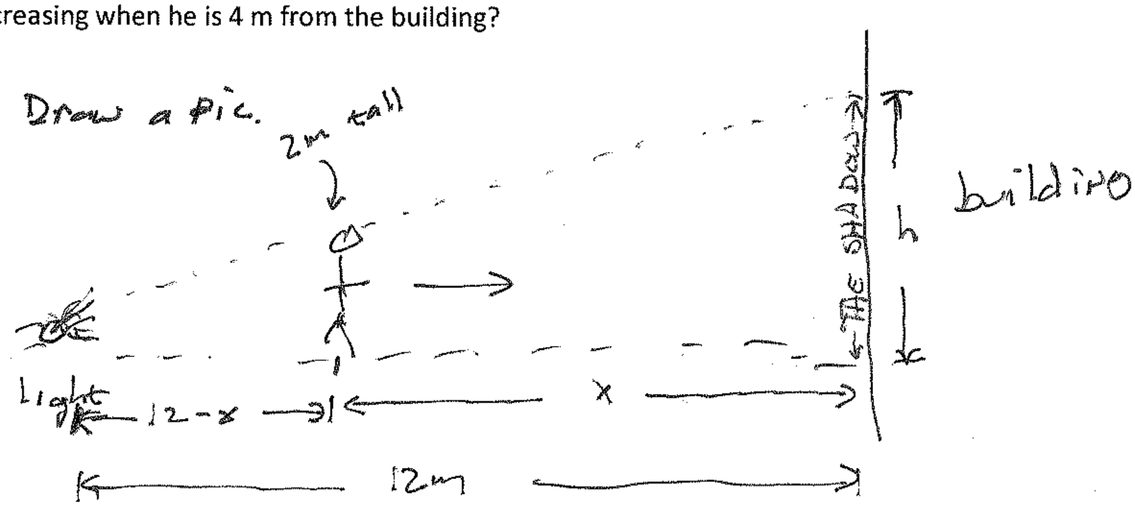
$c = \sqrt{65}$

$\Rightarrow \frac{dB}{dt} = \frac{\sqrt{65}}{8} \cdot (-1)$ m/s

(6) The boat is approaching at $\frac{\sqrt{65}}{8}$ m/s.

Example 2: A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

(1) Draw a pic.



(2) relate the variables: $\frac{2}{12-x} = \frac{h}{12}$

similar triangles

(3) we know: $\frac{dx}{dt} = 1.6 \text{ m/s}$

we care when $x = 4$; $h = 3$ (we find $h=3$ using eqn (2).)

(4) we want: $\frac{dh}{dt}$

(5) implicitly differentiate (2).

$$24 = h(12-x) = 12h - xh$$

$$\Rightarrow \frac{d}{dt} 24 = \frac{d}{dt}(12h - xh)$$

$$\Rightarrow 0 = 12\frac{dh}{dt} - \left(\frac{dx}{dt} h + \frac{dh}{dt} x\right)$$

solve for $\frac{dh}{dt}$.

$$\Rightarrow \frac{dx}{dt} h = \frac{dh}{dt}(12 - x)$$

$$\Rightarrow \frac{dh}{dt} = \frac{h}{12-x} \frac{dx}{dt}$$

$\frac{dx}{dt} = 1.6$

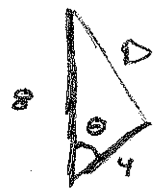
$x = 4$; $h = 3$

$$\frac{dh}{dt} = \frac{3}{8} \cdot 1.6 = \frac{3}{5} \text{ m/s}$$

(6) The shadows he decreases by $\frac{3}{5}$ m/s.

Example 3: The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

(1) pic.



(2) relate variables.

$D^2 = 4^2 + 8^2 - 2(4)(8)\cos\theta$

(the law of cosines).

$\Rightarrow D^2 = 80 - 64\cos\theta$

(3) We know:

$\frac{d\theta}{dt} = \frac{11\pi}{6}$ rad/hr.

we care when $\theta = \frac{\pi}{6}$; $D = \sqrt{80 - 32\sqrt{3}}$

(4) we want:

$\frac{dD}{dt}$

(5) implicit differentiation.

$\Rightarrow \frac{d}{dt} D^2 = \frac{d}{dt}(80 - 64\cos\theta)$

$\Rightarrow 2D \frac{dD}{dt} = +64\sin\theta \cdot \frac{d\theta}{dt}$

$\Rightarrow \frac{dD}{dt} = \frac{64}{2D}\sin\theta \cdot \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = \frac{11\pi}{6}$

$D = \text{ugly}$

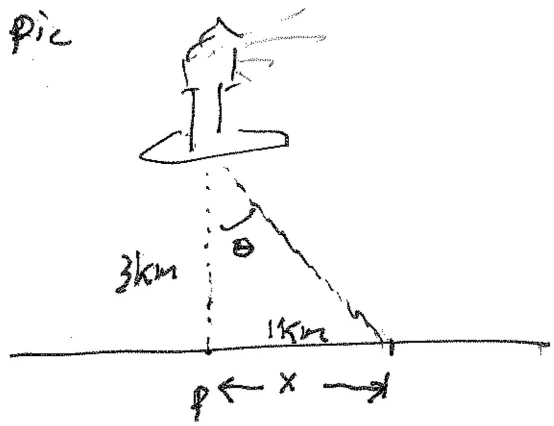
$\theta = \pi/6$

$= \frac{64}{2\sqrt{80-32\sqrt{3}}} \cdot \frac{1}{2} \cdot \frac{11\pi}{6}$ mm/hr

(6) The hands approach at a rate of $\frac{8 \cdot 11\pi}{3\sqrt{80-32\sqrt{3}}}$ mm/hr.

Example 4: A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

(1) pic



(2) relate variables.

$$\tan\theta = \frac{x}{3} \quad \text{(use trig).}$$

(3) we know:

$$\frac{d\theta}{dt} = 8\pi \text{ rad/min.}$$

we care when $x = 1$

$$\theta = \tan^{-1}\left(\tfrac{1}{3}\right)$$

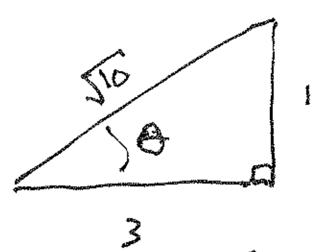
(4) we want $\frac{dx}{dt}$

(5) implicit diff. of (2.).

$$\Rightarrow \frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{3}$$

$$\Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{3}\cdot\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = 3\sec^2\theta\,\frac{d\theta}{dt}\Bigg|_{\frac{d\theta}{dt} = 8\pi,\ x = 1;\ \theta = \tan^{-1}\left(\frac{1}{3}\right)}$$



$$\sec\theta = \frac{\sqrt{10}}{3}$$

$$\Rightarrow \frac{dx}{dt} = 3\cdot\sec^2\Big(\underbrace{\tan^{-1}\left(\tfrac{1}{3}\right)}_{\theta}\Big)\cdot 8\pi$$

$$= 3\cdot\left(\frac{\sqrt{10}}{3}\right)^2\cdot 8\pi$$

$$= \frac{80\pi}{3} \text{ km/min.}$$

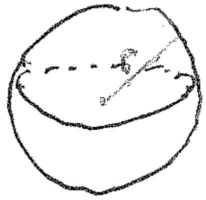
(6) The light moves along the shoreline at $\frac{80\pi}{3}$ km/min.

Example 5: A large spherical balloon is blown up at a constant rate 3 liters every 8 seconds. How fast is the surface area changing when the diameter is 50 cm?

(1) pic

I have three solns to this example thanks to Adam Gutierrez (soln 2) and Kha Nguyen (soln 3)

soln 1:



(2) relate variables

$S = 4\pi r^2$ where $r = \frac{d}{2}$ Also $V = \frac{4}{3}\pi r^3$

$\Rightarrow S = 4\pi\left(\frac{d}{2}\right)^2$ $\Rightarrow V = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3$

$\Rightarrow S = \pi d^2$. solve for d. $\Rightarrow V = \frac{1}{6}\pi d^3$

$d = \sqrt{\frac{S}{\pi}}$ $\Rightarrow V = \frac{1}{6}\pi\left(\frac{S}{\pi}\right)^{3/2}$

(3) we know.

$\frac{dV}{dt} = \frac{3}{8} L/s$

we care when $d = 50cm$

$S = 2500\pi$

(4) we want

$\frac{dS}{dt}$

(5) implicit diff of (2).

$\Rightarrow \frac{d}{dt} V = \frac{d}{dt} \frac{1}{6\sqrt{\pi}} S^{3/2}$

$\Rightarrow \frac{dV}{dt} = \frac{1}{6\sqrt{\pi}} \cdot \frac{3}{2} S^{1/2} \cdot \frac{dS}{dt}$

$\Rightarrow \frac{dS}{dt} = \frac{1}{\frac{1}{6\sqrt{\pi}} \cdot \frac{3}{2}\sqrt{S}} \frac{dV}{dt}$

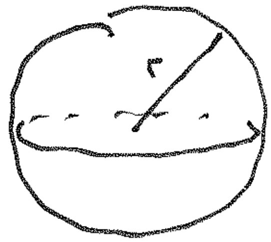
$\frac{dV}{dt} = \frac{3}{8}\frac{L}{s}$

$S = 2500\pi$

$= \frac{6\sqrt{\cancel{\pi}} \cdot 2}{\cancel{3}\sqrt{2500\cancel{\pi}}} \cdot \frac{\cancel{3}}{8}(1000)$ ← since $1000 cm^3 = 1 L$

$= \frac{3(1000)}{2 \cdot 50} = 30 \; cm^2/s$

(6) The surface area of the balloon is growing at $30 \; cm^2/s$.



soln 2:

$$V = \frac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$$

$$S = 4\pi r^2$$

and so $S = 4\pi \left(\frac{3V}{4\pi}\right)^{2/3}$

$$\Rightarrow \ln(S) = \ln\left(4\pi\left(\frac{3V}{4\pi}\right)^{2/3}\right)$$

$$= \ln 4\pi + \frac{2}{3}\left(\ln(3) + \ln(V) - \ln(4\pi)\right)$$

differentiate both sides WRT t.

$$\Rightarrow \frac{S'}{S} = \frac{2}{3}\frac{V'}{V}$$

$$\Rightarrow S' = \frac{2}{3}\frac{S}{V}V'$$

$$= \frac{2}{3}\cdot\frac{4\pi r^2}{\frac{4}{3}\pi r^3}\, V'$$

$$= \frac{2}{3}\cdot\frac{3}{r}\, V'$$

$$= \frac{2}{r}\, V'$$

$$\frac{2}{25}\cdot\frac{3000}{8} = 30\ \frac{cm^2}{s}$$

$r = 25$

$V' = \frac{3000}{8}$

$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$

soln 3: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{8\pi r}\frac{dS}{dt}$$

so $\frac{1}{8\pi r}\frac{dS}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$

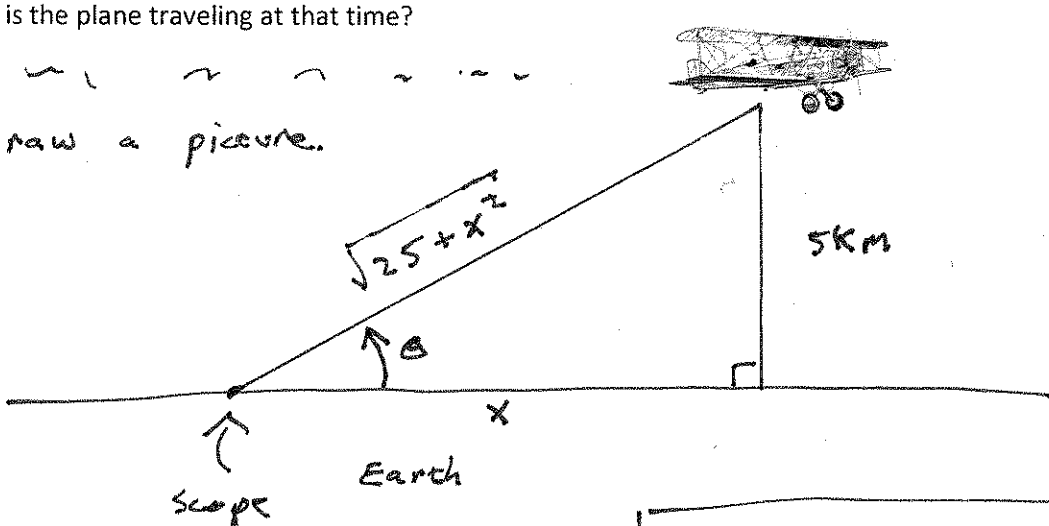
$$\Rightarrow \frac{dS}{dt} = \frac{8\pi r}{4\pi r^2}\frac{dV}{dt}$$

$$\Rightarrow \frac{dS}{dt} = \frac{2}{r}\frac{dV}{dt}\Bigg|_{\substack{r=25 \\ \frac{dV}{dt} = \frac{3000}{8}\frac{cm^3}{s}}} \qquad \frac{2}{25}\cdot\frac{3000}{8}\frac{cm^2}{s}$$

$$30\ \frac{cm^2}{s}$$

Example 6: A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6 \frac{rad}{\min}$. How fast is the plane traveling at that time?

(1) Draw a picture.



Solve for x

$\tan\frac{\pi}{3} = \frac{5}{x}$

$\Rightarrow \sqrt{3} = \frac{5}{x}$

$\Rightarrow x = 5/\sqrt{3}$

(2) relate the variables.

$\tan\theta = \frac{5}{x}$ or $\cot\theta = \frac{x}{5}$

(3) we know $\frac{d\theta}{dt} = -\frac{\pi}{6} \frac{rad}{min}$

(4) we want $\frac{dx}{dt}$

(5) implicit differentiation.

$\frac{d}{dt}\cot\theta = \frac{d}{dt}\frac{x}{5}$

$\Rightarrow -\csc^2\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = -5\csc^2\theta \frac{d\theta}{dt}\Big|_{\substack{\theta = \pi/3 \\ \frac{d\theta}{dt} = -\frac{\pi}{6}}} \quad -5\left(\frac{2}{\sqrt{3}}\right)^2 \cdot -\frac{\pi}{6}$

$= \frac{20\pi}{18}$

$= \frac{10\pi}{9}$

Alternate step (5)

$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{5}{x}$

$\Rightarrow \sec^2\theta \frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = -\frac{x^2\sec^2\theta}{5}\frac{d\theta}{dt}$

at $\theta = \frac{\pi}{3}$ and $\frac{d\theta}{dt} = \frac{-\pi}{6}$ and $x = 5\sqrt{3}$

$\Rightarrow \frac{dx}{dt} = \frac{-(5/\sqrt{3})^2 \cdot 2^2}{5}\left(\frac{-\pi}{6}\right)$

$= \frac{\cancel{5} \cdot 5 \cdot 4\pi}{\cancel{5} \cdot 3 \cdot 6}$

$= 10\pi/9$

The plane is traveling away from the scope at $\frac{10\pi}{9} \frac{Km}{min}$.