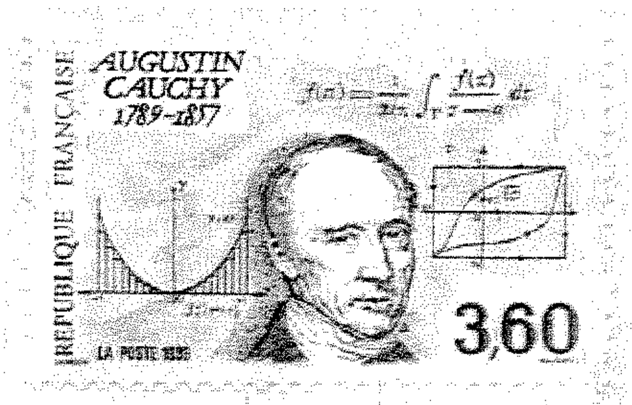
2.4: The Precise Definition of the Limit



Augustin Louis Cauchy
1789 - 1857

Augustin-Louis Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.

Karl Theodor Wilhelm Weierstrass
1815 - 1897

Karl Weierstrass is best known for his construction of the theory of complex functions by means of power series.

<u>Definition</u>: Let f be a function defined on some open interval that contains $x = a$, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L and we write $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Notation:

1.) δ is the lower case Greek letter delta and refers to a small change in x.

2.) ε is the lower case Greek letter epsilon and refers to a small change in y.

Generally, a $\delta - \varepsilon$ proof has two parts:

a) guess @ δ ← a credit

b) prove it works ← \$

Definition: Let f be a function defined on some open interval that contains $x = a$, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L and we write $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\underbrace{0 < |x-a| < \delta}_{(1.)}$ then $\underbrace{|f(x) - L| < \varepsilon}_{(2.)}$.

Example 1: Prove $\lim_{x \to 2} (4x-7) = 1$

a) Guess δ

suppose $|f(x) - L| < \varepsilon$

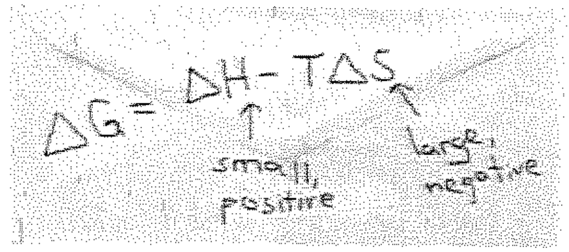
$\Rightarrow |(4x-7) - 1| < \varepsilon$

$\Rightarrow |4x - 8| < \varepsilon$

$\Rightarrow 4|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{4}$

choose $\delta = \frac{\varepsilon}{4}$



Back of the envelope calculation

For (a.), start with (2.) and end with (1.)

b) The proof.

For (b.), start with (1.) and end with (2.)

Claim: $\lim_{x \to 2} (4x - 7) = 1$

proof.

Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{4}$.

If $0 < |x-a| < \delta$

$\Rightarrow 0 < |x - 2| < \frac{\varepsilon}{4}$

$\Rightarrow |x-2| < \frac{\varepsilon}{4}$

$\Rightarrow 4|x-2| < \varepsilon$

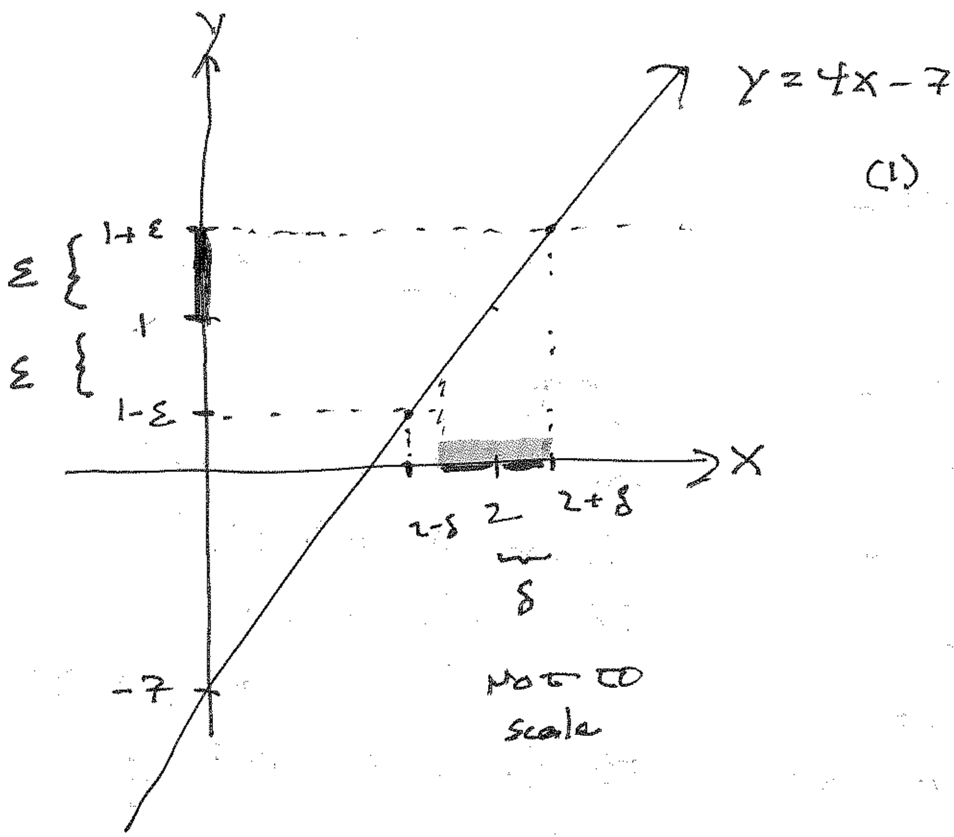
$\Rightarrow |4x - 8| < \varepsilon$

$\Rightarrow |(4x-7) - 1| < \varepsilon$

$\Rightarrow |f(x) - L| < \varepsilon$

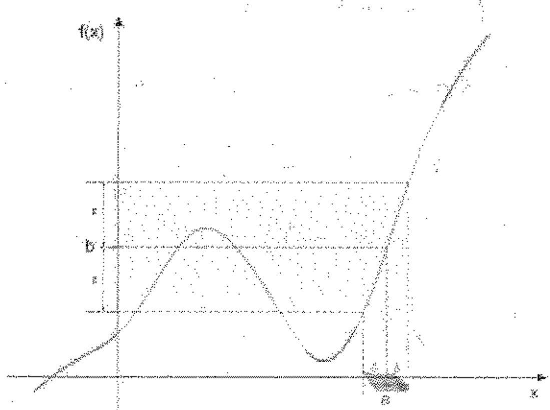
Hence $\lim_{x \to 2} (4x - 7) = 1$.

Example 1 revisited: $\lim_{x \to 2} (4x-7) = 1$

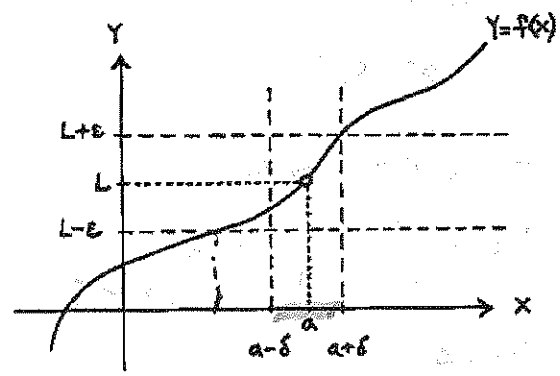


(1) Epsilon: $\varepsilon > 0$ given

Two more general $\delta - \varepsilon$ diagrams. Remember, we choose ε and then use it to determine a corresponding δ.



In this diagram, the emphasis is on the fact that δ represents the width of an interval along the x-axis while ε represents the height of an interval along the y-axis



In this diagram, the emphasis is on the fact that $x = a \pm \delta$ are the vertical boundary lines of a region while $y = b \pm \varepsilon$ are the horizontal boundaries.

Recall: We say that the limit of $f(x)$ as x approaches a is L and we write $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\underbrace{0 < |x-a| < \delta}_{(1.)}$ then $\underbrace{|f(x) - L| < \varepsilon}_{(2.)}$.

Example 2: Prove $\lim_{x \to 2}(x^2 + x - 1) = 5$

a) Guess δ start w/ (2.)

(2.) $|f(x) - L| < \varepsilon$

$\Rightarrow |(x^2 + x - 1) - 5| < \varepsilon$

$\Rightarrow |x^2 + x - 6| < \varepsilon$

$\Rightarrow |(x+3)(x-2)| < \varepsilon$

$\Rightarrow |x+3||x-2| < \varepsilon$

we care about x-vals near $x = 2$.

we will force $|x-2| < 1$

$\Rightarrow -1 < x - 2 < 1$

$\Rightarrow 1 < x < 3$

$\Rightarrow 4 < x + 3 < 6$

$\Rightarrow \underline{|x+3|} < 6$ when $1 < x < 3$

let's force

$\underline{|x+3|}|x-2| < \underline{6}|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{6}$

Guess: $\delta = \min\{1, \frac{\varepsilon}{6}\}$

b) Prove it.

Claim: $\lim_{x \to 2}(x^2 + x - 1) = 5$

Proof.

Let $\varepsilon > 0$ be given.

choose $\delta = \min\{1, \frac{\varepsilon}{6}\}$.

If $0 < |x-a| < \delta$

$\Rightarrow 0 < |x-2| < \min\{1, \frac{\varepsilon}{6}\}$

$\Rightarrow |x-2| < \min\{1, \frac{\varepsilon}{6}\}$

$\Rightarrow |x-2| < 1$ and $|x-2| < \frac{\varepsilon}{6}$

$\Rightarrow 1 < x < 3$

$\Rightarrow 4 < x + 3 < 6$

$\Rightarrow |x+3| < 6$ when $1 < x < 3$.

$\Rightarrow |x+3||x-2| < 6 \cdot \frac{\varepsilon}{6}$

$\Rightarrow |(x^2 + x - 1) - 5| < \varepsilon$

$\Rightarrow |f(x) - L| < \varepsilon$

Hence $\lim_{x \to 2}(x^2 + x - 1) = 5$.

We can use the precise definition to prove the validity of the limit laws in the previous section. The proof of the sum law is given in the text.

Example 3: Prove that if $\underbrace{\lim_{x\to a} f(x) = L}_{(1.)}$ and $\underbrace{\lim_{x\to a} g(x) = M}_{(2.)}$, then $\underbrace{\lim_{x\to a}\left(f(x) - g(x)\right) = L - M}_{(3.)}$

provided the limits exist. This is called the difference law.

a) Guess δ

We want $\left|\left(f(x)-g(x)\right)-(L-M)\right| < \varepsilon$.

We can do this, by requiring $\left|f(x)-L\right| < \frac{\varepsilon}{2}$ and $\left|g(x)-M\right| < \frac{\varepsilon}{2}$.

Since limits (1.) and (2.) exist, given $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then

$\left|f(x)-L\right| < \frac{\varepsilon}{2}$ and there exists a $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $\left|f(x)-L\right| < \frac{\varepsilon}{2}$.

So we will choose $\delta = \min\{\delta_1, \delta_2\}$

b) Prove it.

proof:

Let $\varepsilon > 0$ be given. Choose $\delta = \min\{\delta_1, \delta_2\}$ where

$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$ and $0 < |x-a| < \delta_2 \Rightarrow$

$|g(x) - M| < \frac{\varepsilon}{2}$.

If $0 < |x-a| < \delta \Rightarrow 0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$.

$\Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$

$\Rightarrow |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow |f(x) - L| + |M - g(x)| < \varepsilon$

$\Rightarrow |(f(x) - L) + (M - g(x))| < \varepsilon$ (by the triangle inequality)

$\Rightarrow |(f(x) - g(x)) - (L - M)| < \varepsilon$

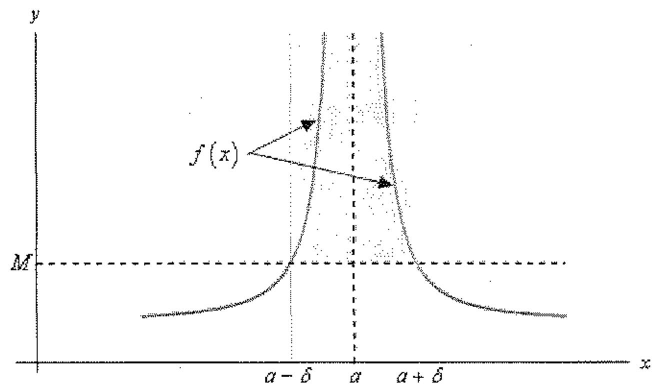
Hence the difference law is proved.

Note: The previous exercise makes use of the **triangle inequality** which states $|a+b| < |a| + |b|$

Definition (an infinite limit): Let f be a function defined on some open interval that contains $x = a$ (except possibly a itself). Then $\lim_{x \to a} f(x) = \infty$ if for all $M > 0$ there exists $\delta > 0$ such that if

$$\underbrace{0 < |x - a| < \delta}_{(1.)} \text{ then } \underbrace{f(x) > M}_{(2.)}$$

In the diagram, notice that when the x-values are within δ of a, then the y-values are all above M.



Example 4 (if time permits): Prove $\lim_{x \to 5} \frac{1}{(x-5)^4} = \infty$

proof.

Let $M > 0$ be given.

Choose $\delta = \sqrt[4]{1/M}$

If $0 < |x-5| < \delta$

$\Rightarrow |x-5| < \sqrt[4]{1/M}$

$\Rightarrow (x-5)^4 < \frac{1}{M}$

$\Rightarrow M < \frac{1}{(x-5)^4}$

Hence $\lim_{x \to 5} \frac{1}{(x-5)^4} = \infty$

Additional Resources:

There is a nice series of lectures on the precise definition of the limit at the Khan Academy. The first is at: http://www.khanacademy.org/video/limit-intuition-review

For more, there were over 300 hits on YouTube for, "precise definition of a limit quadratic"