Test 3
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Math 220

No work = no credit

Name: KEY

Thus, the task is, not so much to see what no one has yet seen; but to think what nobody has yet thought, about that which everybody sees.

Erwin Rudolf Josef Alexander Schrödinger
1887 – 1961 (Austrian physicist)

Warm-ups (1 pt each)[1]: $\vec{e}_1^T \vec{e}_1 =$ 1 $\quad -1^2 =$ -1 $\quad \vec{e}_1 \vec{e}_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

1.) (1 pt) ~~According to Hilbert, how much transcendent or intrinsic meaning is there in mathematics? (See above).~~ Answer using complete English sentences.

The idea is to have new ideas about what is right in front of you.

2.) (10 pts) Consider the rotation-scaling matrix $B = \begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix}$. Find the angle of rotation (degrees or radians) and the scaling factor.

$\frac{\cos\theta}{\sin\theta} = \frac{5}{12} \Rightarrow \theta = \arctan\left(\frac{12}{5}\right) <$ 67.4° / 1.18 rad.

scaling factor: $\sqrt{5^2 + 12^2} = 13$

3.) (12 pts) Define an orthonormal basis for a subspace V.

m L.I vectors that span V, have length 1, and that are orthogonal.

[1] In the warm-ups, $\vec{e}_i$ refers to the standard basis vector in $\mathbb{R}^2$.

4.) (10 pts) Show $A = \begin{bmatrix} 41 & -60 \\ 24 & -31 \end{bmatrix}$ is similar to the rotation-scaling matrix $B = \begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix}$.

Eigenvalues $\lambda = 5 \pm 12i$

Find an eigenvector

$$A - (5 + 12i)I = \begin{bmatrix} 36 - 12i & -60 \\ 24 & -36 - 12i \end{bmatrix} \quad \frac{1}{36 - 12i}$$

row reduce.

$$\begin{bmatrix} 1 & -\frac{3 - i}{2} \\ 24 & -36 - 12i \end{bmatrix} \quad R_2 - 24R_1 \to R_2$$

$$\begin{bmatrix} 1 & -\frac{3 - i}{2} \\ 0 & 0 \end{bmatrix}$$

eigenvector $\begin{bmatrix} \frac{3 + i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} + i \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$\vec{v} + i\vec{w}$

$$S = \begin{bmatrix} 1/2 & 3/2 \\ 0 & 1 \end{bmatrix}$$

and $AS = SB$

5.) (10 pts) Decide if the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ is diagonalizable. If possible, find an invertible S and a diagonal D such that $S^{-1}AS = D$

$\lambda = 1, 2$

find eigenvecs.

$\lambda = 1: \quad A - 1\lambda = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \Rightarrow$ eigenvec $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 2: \quad A - 2\lambda = \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow$ eigenvec $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\Rightarrow S = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

6.) (6 pts) True or False

a.) The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity.

False ... this is backward.

b.) If 1 is the only eigenvalue of an nxn matrix A, then A must be I_n

False. Example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

c.) If $\det(A) = \det(A^T)$ then A must be a symmetric matrix.

False. Example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

7.) (10 pts) Determine whether the zero state of $A=\begin{bmatrix} 2.4 & -2.5 \\ 1 & -0.6 \end{bmatrix}$ is a stable equilibrium of the dynamical system $\vec{x}(t)=A^t\vec{x}_0$.

Find eigenvalues

$\det(A-\lambda I)=0$

$\Rightarrow$ solve $\begin{vmatrix} 2.4-\lambda & -2.5 \\ 1 & -0.6-\lambda \end{vmatrix}=0$

$\Rightarrow (2.4-\lambda)(-0.6-\lambda)+2.5=0$

$\Rightarrow \lambda^2-1.8\lambda-1.44+2.5=0$

$\Rightarrow \lambda^2-1.8\lambda+1.06=0$

$\lambda=\dfrac{1.8\pm\sqrt{1.8^2-4(1)(1.06)}}{2(1)}$

$=\dfrac{1.8\pm\sqrt{-1}}{2}$

$\Rightarrow \lambda=\frac{1}{2}(1.8\pm i)$

and $|\lambda|=\sqrt{.9^2+.5^2}>1$

so the zero state is unstable.

8.) (10 pts) Find the orthogonal projection of $\vec{x}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ onto the subspace with orthonormal basis

$\vec{u}_1=\frac{1}{13}\begin{bmatrix}3\\4\\12\end{bmatrix}$ and $\vec{u}_2=\frac{1}{5}\begin{bmatrix}-4\\3\\0\end{bmatrix}$

$\vec{x}^{\parallel}=(\vec{x}\cdot\vec{u}_1)\vec{u}_1+(\vec{x}\cdot\vec{u}_2)\vec{u}_2$

$=\frac{3}{13}\begin{bmatrix}3/13\\4/13\\12/13\end{bmatrix}+\frac{-4}{5}\begin{bmatrix}-4/5\\3/5\\0\end{bmatrix}$

$=\begin{bmatrix}2929/4225\\-1728/4225\\36/169\end{bmatrix}$

9.) (10 pts) Prove that for every vector $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$ we can write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ where $\vec{x}^{\parallel}$ is in V and $\vec{x}^{\perp}$ is perpendicular to V.

proof.

Let $\vec{x} \in \mathbb{R}^n$ be given and $\vec{u}_1, \dots, \vec{u}_m$ be an orthonormal basis of V.

If $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp} \Rightarrow \vec{x}^{\parallel} = \vec{x} - \vec{x}^{\perp}$

where $\vec{x}^{\parallel} = c_1\vec{u}_1 + \dots + c_m\vec{u}_m$ for yet to be determined $c_1, \dots, c_m$.

$\Rightarrow \vec{x} - \vec{x}^{\perp} = c_1\vec{u}_1 + \dots + c_m\vec{u}_m$

$\Rightarrow \vec{u}_i \cdot (\vec{x} - \vec{x}^{\perp}) = (\vec{u}_i \cdot \vec{x}) = u_i \cdot (c_1\vec{u}_1 + \dots + c_m\vec{u}_m) = c_i$

for $i = 1, \dots, m$

$\Rightarrow$ we have a unique formula for $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$ and our claim is proved. QED.