Test 1 – Part A
Dusty Wilson
Math 220

Name: Key

An elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline
1908-1992 (American mathematician)

No work = no credit
No Graphing Calculators

Warm-ups (1 pt each): $\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\underline{[11]}$ $\quad I_2^{-1}=\underline{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}$ $\quad \vec{e}_1\cdot\vec{e}_1=\underline{1}$

1.) (1 pts) According to Kline (above), how should a good proof be written? Answer using complete English sentences.

A good poem is like a bad poem.

2.) (8 pts) Consider $A=\begin{bmatrix}1 & 2 & -1\\3 & 7 & -10\\7 & 16 & -21\end{bmatrix}$. Find A^{-1} if it exists. If it doesn't exist, write the letters of my first name in alphabetical order.

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\3 & 7 & -10 & 0 & 1 & 0\\7 & 16 & -21 & 0 & 0 & 1\end{array}\right]\begin{array}{l}\\R_2-3R_1\to R_2\\R_3-7R_1\to R_3\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\0 & 1 & -7 & -3 & 1 & 0\\0 & 2 & -14 & -7 & 0 & 1\end{array}\right]\begin{array}{l}\\ \\R_3-2R_2\to R_3\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\0 & 1 & -7 & -3 & 1 & 0\\0 & 0 & 0 & -1 & -2 & 1\end{array}\right]$$

can't get I

DSTUY

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 2 & 6 \\ 5 & -2 & 1 & 8 & 3 \\ -1 & 3 & 2 & 2 & 1 \end{array}\right] \quad R_1 \longleftrightarrow -R_3$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & -2 & -1 \\ 5 & -2 & 1 & 8 & 3 \\ 2 & 1 & 1 & 2 & 6 \end{array}\right] \quad \begin{array}{l} R_2 - 5R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & -2 & -1 \\ 0 & 13 & 11 & 18 & 8 \\ 0 & 7 & 5 & 6 & 8 \end{array}\right] \quad 2R_3 - R_2 \to R_2$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & -2 & -1 \\ 0 & 1 & -1 & -6 & 8 \\ 0 & 7 & 5 & 6 & 8 \end{array}\right] \quad \begin{array}{l} R_1 + 3R_2 \to R_1 \\ R_3 - 7R_2 \to R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & -20 & 23 \\ 0 & 1 & -1 & -6 & 8 \\ 0 & 0 & 12 & 48 & -48 \end{array}\right] \quad \tfrac{1}{12} R_3 \to R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & -20 & 23 \\ 0 & 1 & -1 & -6 & 8 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right] \quad \begin{array}{l} R_1 + 5R_3 \to R_1 \\ R_1 + R_3 \to R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right]$$

$x_1 = 3$

$x_2 = 4 + 2x^4$

$x_3 = -4 - 4x^4$

$x_4 = x^4$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

Test 1 – Part B
Dusty Wilson
Math 220

Name: ______________________________

No work = no credit

1.) (4 pts) If the rank of a 5x3 matrix A is 3, what is rref(A)?

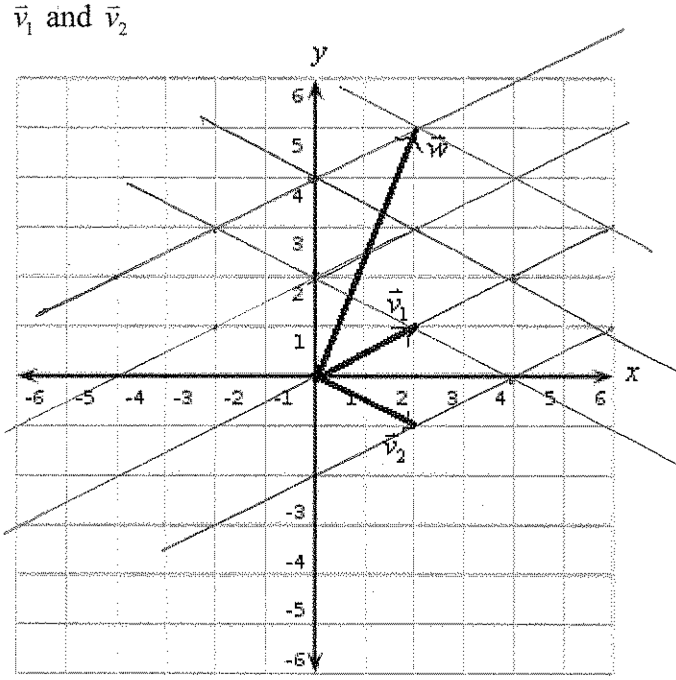
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4pts)
2.) Write $1\begin{bmatrix}4\\5\end{bmatrix}+2\begin{bmatrix}6\\7\end{bmatrix}+3\begin{bmatrix}8\\9\end{bmatrix}$ as a product.

$$\begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

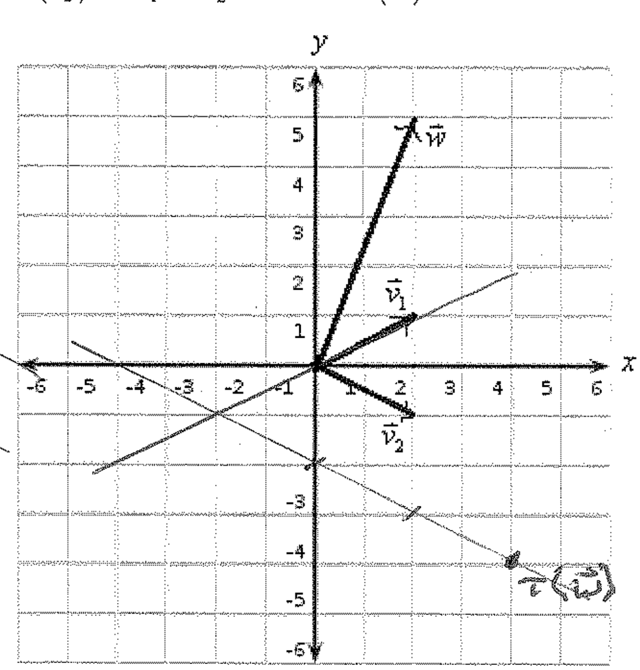
3.) (8 pts) Answer the following:

(a.) Express $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$



$\vec{w} = 3\vec{v}_1 + -2\vec{v}_2$

(b.) Consider a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{v}_1) = \vec{v}_1 + 3\vec{v}_2$ and $T(\vec{v}_2) = 2\vec{v}_1 + 3\vec{v}_2$. Sketch $T(\vec{w})$



$T(\vec{w}) = T(3\vec{v}_1 - 2\vec{v}_2)$
$= 3\,T(\vec{v}_1) - 2\,T(\vec{v}_2)$
$= 3(\vec{v}_1 + 3\vec{v}_2) - 2(2\vec{v}_1 + 3\vec{v}_2)$
$= -\vec{v}_1 + 3\vec{v}_2$

4.) (6 pts) Answer the following. It may help to find an example to justify your answer.

a.) True or False: If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.

$$A^2B = AAB = ABA = BAA = BA^2$$

True.

b.) True or False: There exists a 2x3 matrix A and a 3x2 matrix B such that $AB = I_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

True.

c.) True or False: There exists an invertible 10x10 matrix that has 92 ones among its entries.

False : 2 identical rows

5.) (8 pts) Consider the circular face. For the matrix $A = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}$ draw a sketch showing the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ on this face.

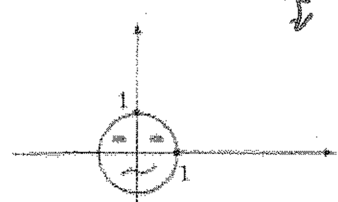
rotate $\frac{\pi}{2}$ ccw

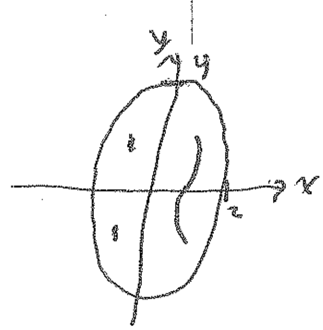
$\leftrightarrow$ by 4

$\updownarrow$ by 2

$$A\vec{e}_1 = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$A\vec{e}_2 = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$





alternate soln.

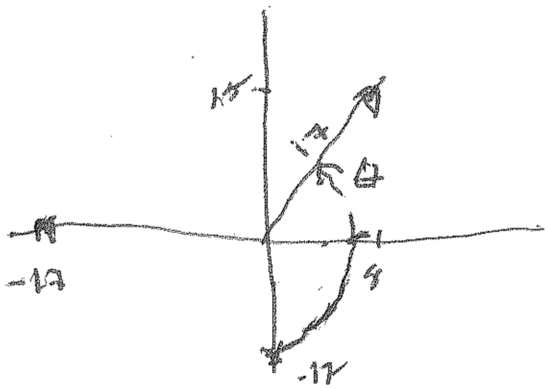
Form of a rotation matrix $A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 0 \\ -17 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$

$\Rightarrow \begin{array}{l} 17s = 8 \Rightarrow s = 8/17 \\ -17c = 15 \Rightarrow c = -15/17 \end{array} \Rightarrow A = \begin{bmatrix} -15/17 & -8/17 \\ 8/17 & -15/17 \end{bmatrix}$

6.) (8 pts) Find a rotation matrix A that transforms $\begin{bmatrix} 0 \\ -17 \end{bmatrix}$ into $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$.

$\theta = \tan^{-1}\left(\frac{15}{8}\right)$



$$A = \begin{bmatrix} \cos\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{15}{8}\right)\right) & -\sin\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{15}{8}\right)\right) \\ \sin\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{15}{8}\right)\right) & \cos\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{15}{8}\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} -15/17 & -8/17 \\ 8/17 & -15/17 \end{bmatrix}$$

7.) (8 pts) Prove that if A is an n x m matrix and $\vec{x}, \vec{y} \in \mathbb{R}^m$, then $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

proof.

Let $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_m \\ | & & | \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ be given.

Now $A(\vec{x}+\vec{y}) = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_m + y_m \end{bmatrix}$

$= (x_1 + y_1)\vec{a}_1 + \dots + (x_m + y_m)\vec{a}_m$

$= (x_1\vec{a}_1 + \dots + x_m\vec{a}_m) + (y_1\vec{a}_1 + \dots + y_m\vec{a}_m)$

$= A\vec{x} + A\vec{y}$

Hence our claim is proved.

8.) (8 pts) Write the product $A\bar{x}$ as a linear combination. Be sure to carefully define A and $\bar{x}$.

$A_{m\times n}$ and $\vec{x} \in \mathbb{R}^n$.

$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

so $A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$.

9.) (8 pts) Suppose $\bar{x} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ and L is the line $L: 21y = 20x$.

pt on line $(21, 20)$

$\vec{u} = \langle \frac{21}{29}, \frac{20}{29} \rangle$

a.) Find the projection of $\bar{x}$ onto L.

$$\vec{x}^{\parallel} = (\vec{x} \cdot \vec{u})\,\vec{u}$$

$$= \frac{283}{29} \langle \frac{21}{29}, \frac{20}{29} \rangle$$

$$\approx \langle 7.07, 6.73 \rangle$$

b.) Find the component of $\bar{x}$ perpendicular to the line L.

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$$

$$= \langle \frac{-3420}{29^2}, \frac{3591}{29^2} \rangle$$

$$\approx \langle -4.07, 4.27 \rangle$$

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Version: 173

$T(v_1) = 1v_1 + 3v_2$ and $T(v_2) = 2v_1 + 3v_2$

$x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Version: 173

$T(v_1) = 1v_1 + 3v_2$ and $T(v_2) = 2v_1 + 3v_2$

$x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(1.) $A = \begin{pmatrix} \frac{9}{2} & -1 \\ -\frac{3}{4} & -\frac{1}{2} \end{pmatrix}$

(2.) $T(x) = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

(3.) $[x]_B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(4.) $B = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}$

(5.) $S\,B = \begin{pmatrix} 8 & 10 \\ -2 & -1 \end{pmatrix}$ and $A\,S = \begin{pmatrix} 8 & 10 \\ -2 & -1 \end{pmatrix}$

Reference: $[T(x)]_B = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Reference: $S = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$

Out[49]=