9:07

Test 3
Dusty Wilson
Math 220

Name: Key

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert
1862 - 1943 (Prussian mathematician)

No work = no credit

No Calculator

Warm-ups (1 pt each)[1]: $\vec{e}_2 \cdot \vec{e}_2 =$ 1 $\quad \vec{e}_1^T \vec{e}_2 =$ [0] $\quad \vec{e}_1 \vec{e}_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

1.) (1 pt) According to Hilbert, how much transcendent or intrinsic meaning is there in mathematics? (See above). Answer using complete English sentences.

Math is w/o intrinsic meaning.

2.) (10 pts) Evaluate $\det(A)$ by hand given $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$

$$\det A = -2\begin{vmatrix} -1 & 2 & -2 \\ 3 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= -2\left(-2 + 4 + 6 - 1 - 12 - (-4)\right)$$

$$+1\left(4 + 4 + 1 - 2 - 2 - 4\right)$$

$$= -2(-1) + 1(1)$$

$$= 3$$

[1] In the warm-ups, $\vec{e}_i$ refers to the standard basis vector in $\mathbb{R}^2$.

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1.) (10 pts) Find the QR factorization of $A = \begin{bmatrix} 12 & 1 \\ 0 & -1 \\ 3 & 1 \\ 4 & -1 \end{bmatrix}$

Stickie Here

$\|\vec{v}_1\| = 13$

$\vec{u}_1 = \begin{bmatrix} 12/13 \\ 0 \\ 3/13 \\ 4/13 \end{bmatrix}$

$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \frac{11}{13}$

$\vec{v}_2^{\perp} = \vec{v}_2 - \frac{11}{13}\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{11}{13}\begin{bmatrix} 12/13 \\ 0 \\ 3/13 \\ 4/13 \end{bmatrix} = \frac{1}{169}\begin{bmatrix} 37 \\ -169 \\ 136 \\ -213 \end{bmatrix}$

$\|\vec{v}_2^{\perp}\| = \frac{1}{169}\sqrt{37^2 + 169^2 + 136^2 + 213^2}$

$= \frac{\sqrt{93795}}{169}$

$\vec{u}_2 = \frac{1}{\sqrt{93795}} \begin{bmatrix} 37 \\ -169 \\ 136 \\ -213 \end{bmatrix}$

.121
-.552
.444
-.695

$A = \begin{bmatrix} 12/13 & 37/\sqrt{93795} \\ 0 & -169/\sqrt{93795} \\ 3/13 & 136/\sqrt{93795} \\ 4/13 & -213/\sqrt{93795} \end{bmatrix} \begin{bmatrix} 13 & \frac{11}{13} \\ 0 & \frac{\sqrt{93795}}{169} \end{bmatrix}$

1.812

2.) (10 pts) Consider the experimental observations given in the following table:

t	-2	-1	0	1	2
y	12	5	3	2	4

Find the least-squares quadratic ($y = at^2 + bt + c$) fit to the data using techniques developed in linear algebra. Give exact (fraction) values in your answer.

$$4a - 2b + c = 12$$
$$a - b + c = 5$$
$$c = 3$$
$$a + b + c = 2$$
$$4a + 2b + c = 4$$

$$\longrightarrow \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 19/14 \\ -19/10 \\ 87/35 \end{bmatrix} = x^*$$

$$y = \frac{19}{14} t^2 - \frac{19}{10} t + \frac{87}{35}$$

$\frac{6}{10}$ for linear

$y = -1.9t + 5.2$

-1 if no frac.

Find the magnitude of the minimum error vector.

$$\text{error} = b - Ax^* = \begin{bmatrix} 2/7 & \frac{-26}{35} & \frac{18}{35} & \frac{2}{35} & \frac{-4}{35} \end{bmatrix}^T$$

$$\|\text{error}\| = \sqrt{\frac{32}{35}} \approx 0.956$$

3.) (2 pts) A matrix A such that $A^T A = I$ is said to be an orthogonal matrix.

4.) (2 pts) True or False: The equation $\det(-A) = \det(A)$ for all 6x6 matrices.

six sign changes.

5.) (2 pts) True or False: The determinant of any diagonal nxn matrix is the product of its diagonal entries.

6.) (2 pts) True or False: If A is an invertible nxn matrix, then $\det(A^T) = \det(A^{-1})$.

No ... what if $A = [2]$.

7.) (5 pts) Prove that if A is invertible, then $\det(A^{-1}) = 1/\det(A)$

proof.

Suppose A is invertible.

$\Rightarrow A^{-1} \cdot A = I$

$\Rightarrow \det(A^{-1}A) = 1$

$\Rightarrow \det(A^{-1})\det(A) = 1$

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$.

Q.E.D.

2/5 for 2x2

8.) (10 pts) Suppose V is the subspace spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ -7 \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$, do the following:

a.) Show $\vec{x}$ is not in V.

$\operatorname{rref}\left(\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{x} \\ | & | & | \end{bmatrix}\right) = I$ so $\vec{x}$ is L.I. of ~~xxxx~~ $\vec{v}_1$ & $\vec{v}_2$ and so not in V.

b.) Find $x^{\parallel}$ $\vec{u}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ & $\vec{u}_2 = \frac{1}{\sqrt{90}} \cdot \begin{bmatrix} 4 \\ 5 \\ -7 \end{bmatrix}$

$$x^{\parallel} = \underbrace{(\vec{u}_1 \cdot \vec{x})}_{4.\bar{3}}\vec{u}_1 + \underbrace{(\vec{u}_2 \cdot \vec{x})}_{-1.16}\vec{u}_2 = \begin{bmatrix} 0.955 \\ 2.278 \\ 3.745 \end{bmatrix}$$

c.) Find $x^{\perp} = \vec{x} - \vec{x}^{\parallel} = \begin{bmatrix} 2.045 \\ -1.278 \\ 0.255 \end{bmatrix}$

9.) (5 pts) Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose for two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^2$ we have $T(\vec{v}_1) = 3\vec{v}_1$ and $T(\vec{v}_2) = 5\vec{v}_2$. What can you say about $\det(A)$? Justify your answer carefully.

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, bend right, "S^{-1}"'] & T(\vec{x}) = Ax \\ {[x]_B} \arrow[u, bend right, "S"'] \arrow[r, "B = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}"'] & {[T(x)]_B} \arrow[u, "S"'] \end{tikzcd}$$

$S = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$

$\det(A) = \det(SBS^{-1})$

$= \det(S)\det(B)\det(S^{-1})$

$= \det(B)$

$= 15.$

10.) (10 pts) Use the determinant to find out for which values of the constant λ the matrix $A - \lambda I$ fails to be invertible. Note: You may find it helpful to factor by grouping.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

$\frac{8}{10}$ if almost right
$\frac{6}{10}$ if close alge
$\frac{5}{10}$ if alg
$\frac{4}{10}$ for 0 @ (2,2)!

$$\cancel{\det(A) = -(-1)\begin{vmatrix} -12 & 5 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} & \\ & \end{vmatrix}}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda)(-1-\lambda) + (-20) + (-24)$$

$$- 4\lambda - (-10)(3-\lambda) - (-1-\lambda)(12)$$

$$= -\lambda(-3 - 2\lambda + \lambda^2) - 44 - 4\lambda + 30 - 10\lambda + 12 + 12\lambda$$

$$= 3\lambda + 2\lambda^2 - \lambda^3 - 2 - 2\lambda$$

$$= (-\lambda^3 + 2\lambda^2) + (\lambda - 2)$$

$$= -\lambda^2(\lambda - 2) + 1(\lambda - 2)$$

$$= (\lambda - 2)(1 - \lambda^2)$$

$$= (\lambda - 2)(1 - \lambda)(1 + \lambda)$$

$\lambda = 2, \pm 1$ when
$\det(A - \lambda I) = 0.$