Test 1
Dusty Wilson
Math 220

Name: KEY

An elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline
1908-1992 (American mathematician)

No work = no credit
No Graphing Calculators

Warm-ups (1 pt each): $A_{nxm} \cdot \vec{0}_m = \underline{\vec{0}_n}$ $\quad A \cdot A^{-1} = \underline{I}$ $\quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$

1.) (1 pts) According to Kline (above), how should a good proof be written? Answer using complete English sentences.

A good proof is like a bad poem.

2.) (8 pts) Consider $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 3 \end{bmatrix}$

a.) Find A^{-1} if it exists. If it doesn't exist, write my middle name backwards.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -4 & 3 & 0 & 0 & 1 \end{array}\right] R_3 + 4R_2 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 4 & 1 \end{array}\right] -R_3 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -4 & -1 \end{array}\right] R_2 + R_3 \to R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & 0 & -4 & -1 \end{array}\right]$$

$R_1 + R_2 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & 0 & -4 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -3 & -1 \\ 0 & -4 & -1 \end{bmatrix}$$

b.) If $b = \begin{bmatrix} -8 \\ -9 \\ -11 \end{bmatrix}$, solve $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & -3 & -1 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} -8 \\ -9 \\ -11 \end{bmatrix} = \begin{bmatrix} 30 \\ 38 \\ 47 \end{bmatrix}$$

3.) (7 pts) Consider the system of linear

$$\begin{aligned} 2x_1 - x_2 - 2x_3 &= 2 \\ 5x_2 - 4x_3 &= -2 \\ x_1 - 3x_2 + x_3 &= 1 \end{aligned}$$

a.) Write the associated coefficient matrix A

$$\begin{bmatrix} 2 & -1 & -2 \\ 0 & 5 & -4 \\ 1 & -3 & 1 \end{bmatrix}$$

b.) Solve the system using Gauss-Jordan Elimination. Write your solution in vector form.

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 2 \\ 0 & 5 & -4 & -2 \\ 1 & -3 & 1 & 1 \end{array}\right] \quad \tfrac{1}{2}R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & -\tfrac{1}{2} & -1 & 1 \\ 0 & 5 & -4 & -2 \\ 1 & -3 & 1 & 1 \end{array}\right] \quad R_3 - R_1 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -\tfrac{1}{2} & -1 & 1 \\ 0 & 5 & -4 & -2 \\ 0 & -\tfrac{5}{2} & 2 & 0 \end{array}\right] \quad \tfrac{1}{5}R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -\tfrac{1}{2} & -1 & 1 \\ 0 & 1 & -\tfrac{4}{5} & -\tfrac{2}{5} \\ 0 & -\tfrac{5}{2} & 2 & 0 \end{array}\right]$$

$R_3 + \tfrac{5}{2}R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -\tfrac{1}{2} & -1 & 1 \\ 0 & 1 & -\tfrac{4}{5} & -\tfrac{2}{5} \\ 0 & 0 & 0 & -1 \end{array}\right]$$

No solution.

c.) What is the rank of the coefficient matrix A found in (a.)?

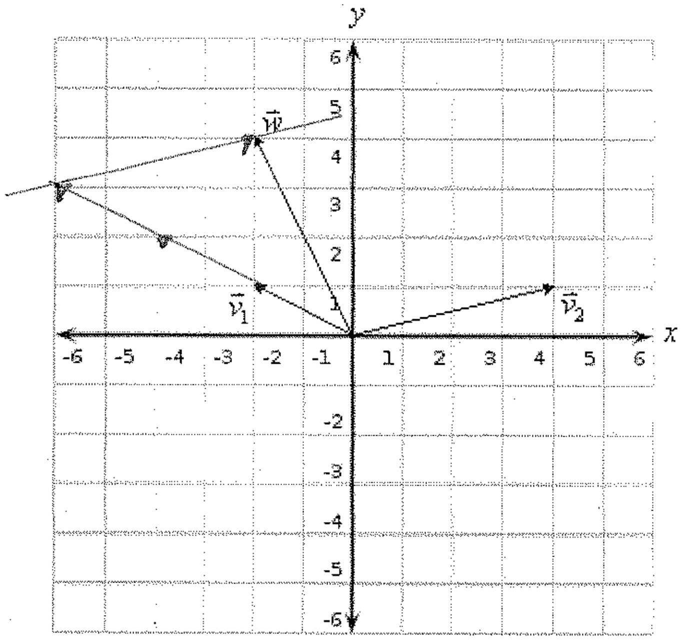
Rank(A) = 2

$T(\vec{w}) = T(3\vec{v}_1 + \vec{v}_2)$
$= 3T(\vec{v}_1) + T(\vec{v}_2)$
$= -2\vec{v}_1 - \vec{v}_2$

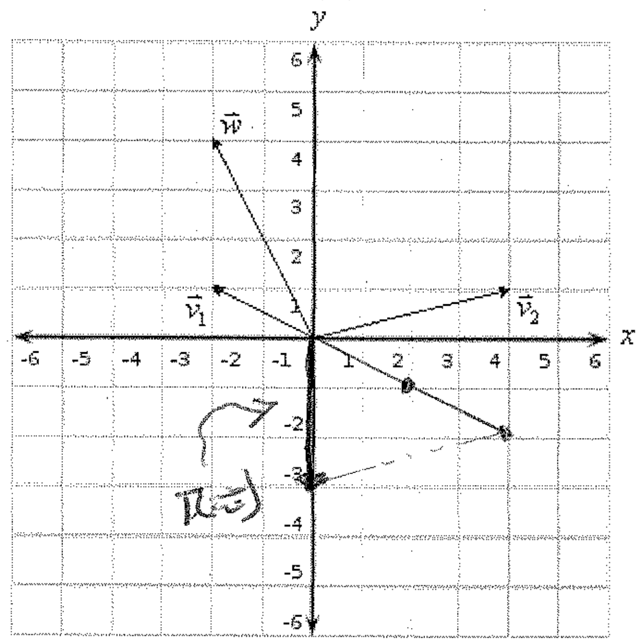
4.) (8 pts) Answer the following:

(a.) Express $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$\vec{w} = 3\vec{v}_1 + \vec{v}_2$



(b.) Consider a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{v}_1) = -\frac{2}{3}v_1$ and $T(\vec{v}_2) = -\vec{v}_2$. Sketch $T(\vec{w})$ on the same axes.



5.) (4 pts) Give an example a geometric linear transformation that has the following property. Your answer should be the name of a transformation, not a specific matrix. (Hint: remember the bug).

a.) An invertible geometric linear transformation

Scale, rotation, reflection, or shear

b.) A non-invertible geometric linear transformation

projection.

6.) (4 pts) A linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^2$ can be written as $T(\vec{x}) = A\vec{x}_4$. What are the dimensions of the matrix A?

$A_{2\times 4}$

7.) (4 pts) Suppose the equation $A_{5\,x\,7}\vec{x}_{7} = \vec{0}$ is solved given that $rank(A) = 3$.

a.) Is it possible for the equation to have "no solution?" Why or why not?

No. since this is a homogeneous eqn. you have 0 in the right col. of rref([A|0]) so will never get 0=1.

b.) How many free variables are there?

4.

8.) (4 pts) Prove that if $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, $\vec{x} \in \mathbb{R}^m$, and k is a scalar, then $T(k\vec{v}) = kT(\vec{v})$.

□ proof.

$$T(k\vec{v}) = A(k\vec{v})$$
$$= kA\vec{v}$$
$$= kT(\vec{v})$$

9.) (4 pts) Consider $x_1\vec{v}_1 + \ldots + x_m\vec{v}_m$ where $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$.

a.) Write the sum as the product of a matrix and a vector. Clearly indicate the contents of the matrix and vector in terms of the x's and v's.

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = A\vec{x}$$

b.) What are the dimensions of the matrix? What are the dimensions of the vector?

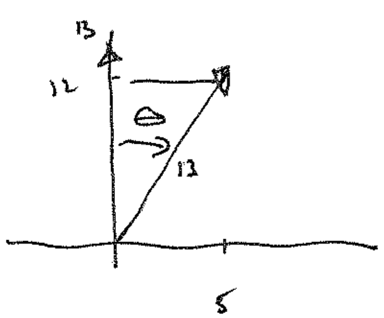
$A_{n \times m}$

$\vec{x}_{m \times 1}$

10.) (4 pts) Find the scaling matrix A that transforms $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ into $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

11.) (4 pts) Find the rotation matrix B that transforms $\begin{bmatrix} 0 \\ 13 \end{bmatrix}$ into $\begin{bmatrix} 5 \\ 12 \end{bmatrix}$



$\cos\theta = \frac{12}{13}$

$\sin\theta = \frac{5}{13}$

rotate C.W...

$$B = \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{-5}{13} & \frac{12}{13} \end{bmatrix}$$