7.4: Diagonalization

The Plan

T : 7.4 Diagonalization	T : 7.5
W : 7.4	W : 7.6 Stability.
Th : 7.5 G	Th : 7.6
F : HW (7.1–4)	F : HW (7.4–6)

Note: I need study group sign-ins.

Focus: Dynamical Systems.

Picture

Coyotes & Road Runners

Symbolic

$\vec{x}(t+1) = A\vec{x}(t)$ w/ $\vec{x}(0) = \vec{x}_0$

$\vec{x}(t)$ = state vector

$\vec{x}_0$ = initial state vector

Graphically

Phase Portrait

Ex 1: Remember our coyote & roadrunner example from (7.1).

$$A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}$$

$$E_{1.1} = \text{span}\left(\begin{bmatrix} 100 \\ 300 \end{bmatrix}\right)$$

$$E_{0.9} = \text{span}\left(\begin{bmatrix} 200 \\ 100 \end{bmatrix}\right)$$

Goal: Find a closed form equation for $\vec{x}(t)$.

$$\vec{x}(t) = c_1 \overbrace{\begin{bmatrix} 100 \\ 300 \end{bmatrix}}^{\vec{v}_1} + c_2 \overbrace{\begin{bmatrix} 200 \\ 100 \end{bmatrix}}^{\vec{v}_2}$$

$$\begin{tikzcd}
\vec{x}(t) \arrow[r, "A"] \arrow[d, "S^{-1}"'] & T(\vec{x}) = A\vec{x} = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = 1.1c_1\vec{v}_1 + 0.9c_2\vec{v}_2 \\
\left[\vec{x}(t)\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \arrow[r, "D = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix}"] & \left[T(\vec{x})\right]_{\mathcal{B}} = \begin{bmatrix} 1.1\,c_1 \\ 0.9\,c_2 \end{bmatrix} \arrow[u, "S"']
\end{tikzcd}$$

where $S = \begin{bmatrix} 100 & 200 \\ 300 & 100 \end{bmatrix}$

So $D = S^{-1} A S$

An $n \times n$ matrix A is called diagonalizable if A is similar to some diagonal matrix D, that is, if there exists an invertible $n \times n$ S s.t. $S^{-1} A S$ is diagonal.

Find a closed form soln. for $\vec{x}(t)$ in the coyote problem. $\vec{x}(t) = c_1 (1.1)^t \vec{v}_1 + c_2 (0.9)^t \vec{v}_2$

ex2: Diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$E_0 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$ and. $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

If $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

then $A = S^{-1} D S$.

Thm: The matrix of a lin. trans. WRT an eigenbasis.

Consider a lin. trans. $T(\vec{x}) = A\vec{x}$ where A is a square matrix. Suppose $D = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an eigenbasis for T w/ $A\vec{v}_i = \lambda_i \vec{v}_i$. Then the D-matrix D of T is

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \text{ where } S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

matrix D is diagonal, and its diagonal entries are the eigenvalues $\lambda_1, \dots, \lambda_n$ of T.

Thm:

(a) Matrix A is diagonalizable iff $\exists$ an eigenbasis for A.

(b) If an $n \times n$ matrix has n distinct eigenvalues, then A is diagonalizable.

Diagonalization Process of $A_{n\times n}$

(a) Find the eigenvals.

(b) Find each eigenspace.

(c) if the sum of $\dim(E_\lambda) \neq n$, stop.

(d) else, construct D & S.

Thm: Powers of a Diagonalizable Matrix.

If A can be diagonalized as $A = SDS^{-1}$

Then $A^t = SD^tS^{-1}$.