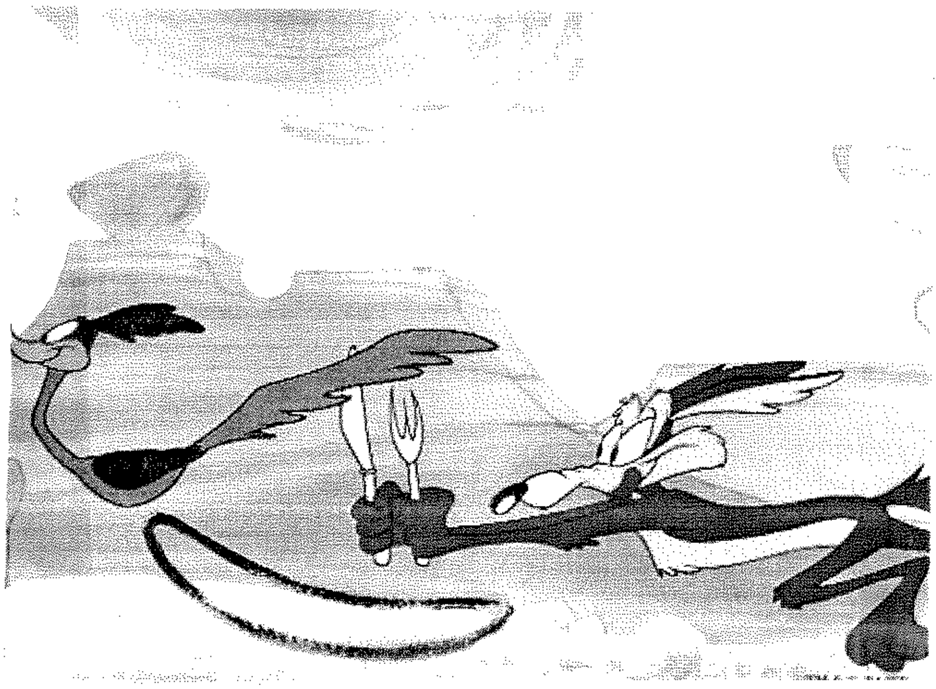
INTRO TO EIGENVECTORS & EIGENVALUES.

A stretch of desert in northwestern Mexico is populated mainly by two species of animals: coyotes and roadrunners. We wish to model the populations $c(t)$ and $r(t)$ of coyotes and roadrunners t years from now if the current populations c_0 and r_0 are known.[1,2]



For this habitat, the following equations model the transformation of this system from one year to the next, from time t to time $(t+1)$:

$$\left| \begin{aligned} c(t+1) &= 0.86c(t) + 0.08r(t) \\ r(t+1) &= -0.12c(t) + 1.14r(t) \end{aligned} \right| .$$

Why is the coefficient of $c(t)$ in the first equation less than 1, while the coefficient of $r(t)$ in the second equation exceeds 1? What is the practical significance of the signs of the other two coefficients, 0.08 and -0.12?

The two equations can be written in matrix form, as

$$\begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix} = \begin{bmatrix} 0.86c(t) + 0.08r(t) \\ -0.12c(t) + 1.14r(t) \end{bmatrix} = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} .$$

The vector

$$\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$$

is called the *state vector* of the system at time t, because it completely describes this system at time t. If we let

$$A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix},$$

we can write the preceding matrix equation more succinctly as

$$\vec{x}(t+1) = A\vec{x}(t).$$

The transformation the system undergoes over the period of one year is linear, represented by the matrix A.

$$\vec{x}(t) \xrightarrow{A} \vec{x}(t+1)$$

Suppose we know the initial state

$$\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix}.$$

We wish to find $\vec{x}(t)$, for an arbitrary positive integer t:

$$\vec{x}(0) \xrightarrow{A} \vec{x}(1) \xrightarrow{A} \vec{x}(2) \xrightarrow{A} \vec{x}(3) \xrightarrow{A} \cdots \xrightarrow{A} \vec{x}(t) \xrightarrow{A} \cdots.$$

We can find $\vec{x}(t)$ by applying the transformation t times to $\vec{x}(0)$:

$$\vec{x}(t) = A^t\vec{x}(0) = A^t\vec{x}_0.$$

Although it is extremely tedious to find $\vec{x}(t)$ with paper and pencil for large t, we can easily compute $\vec{x}(t)$ using technology. For example, given

$$\vec{x}_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix},$$

we find that

$$\vec{x}(10) = A^{10}\vec{x}_0 \approx \begin{bmatrix} 80 \\ 170 \end{bmatrix}.$$

To understand the long-term behavior of this system and how it depends on the initial values, we must go beyond numerical experimentation. It would be useful to have *closed formulas* for $c(t)$ and $r(t)$, expressing these quantities as functions of t. We will first do this for certain (carefully chosen) initial state vectors.

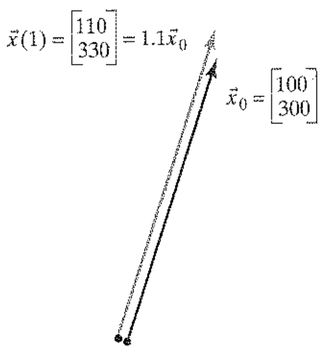


Figure 1

Case 1 ■ Suppose we have $c_0 = 100$ and $r_0 = 300$. Initially, there are 100 coyotes and 300 roadrunners, so that $\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$. Then

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix}.$$

Note that each population has grown by 10%. This means that the state vector $\vec{x}(1)$ is a scalar multiple of $\vec{x}_0$ (see Figure 1):

$$\vec{x}(1) = A\vec{x}_0 = 1.1\vec{x}_0.$$

It is now easy to compute $\vec{x}(t)$ for arbitrary t, using linearity:

$$\begin{aligned} \vec{x}(2) &= A\vec{x}(1) = A(1.1\vec{x}_0) = 1.1A\vec{x}_0 = 1.1^2\vec{x}_0 \\ \vec{x}(3) &= A\vec{x}(2) = A(1.1^2\vec{x}_0) = 1.1^2A\vec{x}_0 = 1.1^3\vec{x}_0 \\ &\vdots \\ \vec{x}(t) &= 1.1^t\vec{x}_0. \end{aligned}$$

We keep multiplying the state vector by 1.1 each time we apply the transformation A.

Recall that our goal is to find closed formulas for $c(t)$ and $r(t)$. We have

$$\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} = 1.1^t\vec{x}_0 = 1.1^t \begin{bmatrix} 100 \\ 300 \end{bmatrix},$$

so that

$$c(t) = 100(1.1)^t \qquad \text{and} \qquad r(t) = 300(1.1)^t.$$

Both populations will grow exponentially, by 10% each year.

Case 2 ■ Suppose we have $c_0 = 200$ and $r_0 = 100$. Then

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = 0.9\vec{x}_0.$$

Both populations decline by 10% in the first year and will therefore decline another 10% each subsequent year. Thus

$$\vec{x}(t) = 0.9^t\vec{x}_0,$$

so that

$$c(t) = 200(0.9)^t \quad \text{and} \quad r(t) = 100(0.9)^t.$$

The initial populations are mismatched: Too many coyotes are chasing too few roadrunners, a bad state of affairs for both species.

Case 3 ■ Suppose we have $c_0 = r_0 = 1{,}000$. Then

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 1{,}000 \\ 1{,}000 \end{bmatrix} = \begin{bmatrix} 940 \\ 1{,}020 \end{bmatrix}.$$

Things are not working out as nicely as in the first two cases we considered: The state vector $\vec{x}(1)$ fails to be a scalar multiple of the initial state $\vec{x}_0$. Just by computing $\vec{x}(2), \vec{x}(3), \dots,$ we could not easily detect a trend that would allow us to generate closed formulas for $c(t)$ and $r(t)$. We have to look for another approach.

The idea is to work with the two vectors

$$\vec{v}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

considered in the first two cases, for which $A^t\vec{v}_i$ was easy to compute. Since the vectors $\vec{v}_1$ and $\vec{v}_2$ form a basis of $\mathbb{R}^2$, any vector in $\mathbb{R}^2$ can be written uniquely as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. This holds in particular for the initial state vector

$$\vec{x}_0 = \begin{bmatrix} 1{,}000 \\ 1{,}000 \end{bmatrix}$$

of the coyote–roadrunner system:

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2.$$

A straightforward computation shows that the coordinates are $c_1 = 2$ and $c_2 = 4$:

$$\vec{x}_0 = 2\vec{v}_1 + 4\vec{v}_2.$$

Recall that $A^t\vec{v}_1 = (1.1)^t\vec{v}_1$ and $A^t\vec{v}_2 = (0.9)^t\vec{v}_2$. Therefore,

$$\begin{aligned} \vec{x}(t) &= A^t\vec{x}_0 = A^t(2\vec{v}_1 + 4\vec{v}_2) = 2A^t\vec{v}_1 + 4A^t\vec{v}_2 \\ &= 2(1.1)^t\vec{v}_1 + 4(0.9)^t\vec{v}_2 \\ &= 2(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}. \end{aligned}$$

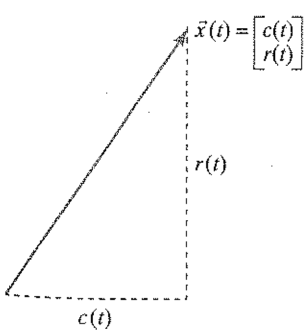


Figure 2

Considering the components of this equation, we can find formulas for $c(t)$ and $r(t)$:

$$\begin{aligned} c(t) &= 200(1.1)^t + 800(0.9)^t \\ r(t) &= 600(1.1)^t + 400(0.9)^t. \end{aligned}$$

Since the terms involving 0.9^t approach zero as t increases, both populations eventually grow by about 10% a year, and their ratio $r(t)/c(t)$ approaches $600/200 = 3$.

Note that the ratio $r(t)/c(t)$ can be interpreted as the slope of the state vector $\vec{x}(t)$, as shown in Figure 2.

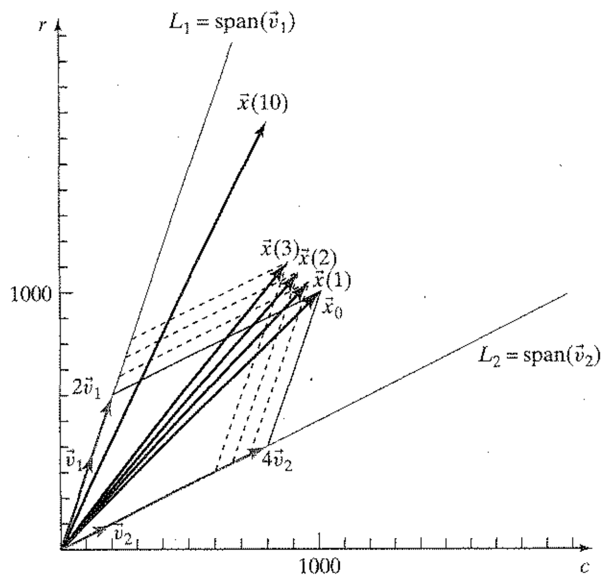


Figure 3

How can we represent the preceding computations graphically?

Figure 3 shows the representation $\vec{x}_0 = 2\vec{v}_1 + 4\vec{v}_2$ of $\vec{x}_0$ as the sum of a vector on $L_1 = \text{span}(\vec{v}_1)$ and a vector on $L_2 = \text{span}(\vec{v}_2)$. The formula

$$\vec{x}(t) = (1.1)^t 2\vec{v}_1 + (0.9)^t 4\vec{v}_2$$

now tells us that the component in L_1 grows by 10% each year, while the component in L_2 shrinks by 10%. The component $(0.9)^t 4\vec{v}_2$ in L_2 approaches $\vec{0}$, which means that the tip of the state vector $\vec{x}(t)$ approaches the line L_1, so that the slope of the state vector will approach 3, the slope of L_1.

To show the evolution of the system more clearly, we can sketch just the endpoints of the state vectors $\vec{x}(t)$. Then the changing state of the system will be traced out as a sequence of points in the c-r plane.

It is natural to connect the dots to create the illusion of a continuous *trajectory*. (Although, of course, we do not know what really happens between times t and $t + 1$.)

Sometimes we are interested in the state of the system in the past, at times $-1, -2, \ldots$. Note that $\vec{x}(0) = A\vec{x}(-1)$, so that $\vec{x}(-1) = A^{-1}\vec{x}_0$ if A is invertible (as in our example). Likewise, $\vec{x}(-t) = (A^t)^{-1}\vec{x}_0$, for $t = 2, 3, \ldots$. The trajectory (future and past) for our coyote–roadrunner system is shown in Figure 4.

To get a feeling for the long-term behavior of this system and how it depends on the initial state, we can draw a rough sketch that shows a number of different trajectories, representing the various qualitative types of behavior. Such a sketch is called a *phase portrait* of the system. In our example, a phase portrait might show the foregoing three cases, as well as a trajectory that starts above L_1 and one that starts below L_2. See Figure 5.

To sketch these trajectories, express the initial state vector $\vec{x}_0$ as the sum of a vector $\vec{w}_1$ on L_1 and a vector $\vec{w}_2$ on L_2. Then see how these two vectors change over time. If $\vec{x}_0 = \vec{w}_1 + \vec{w}_2$, then

$$\vec{x}(t) = (1.1)^t \vec{w}_1 + (0.9)^t \vec{w}_2.$$

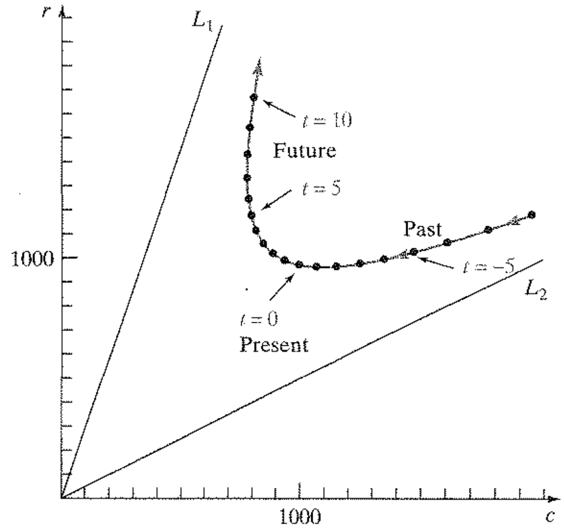


Figure 4

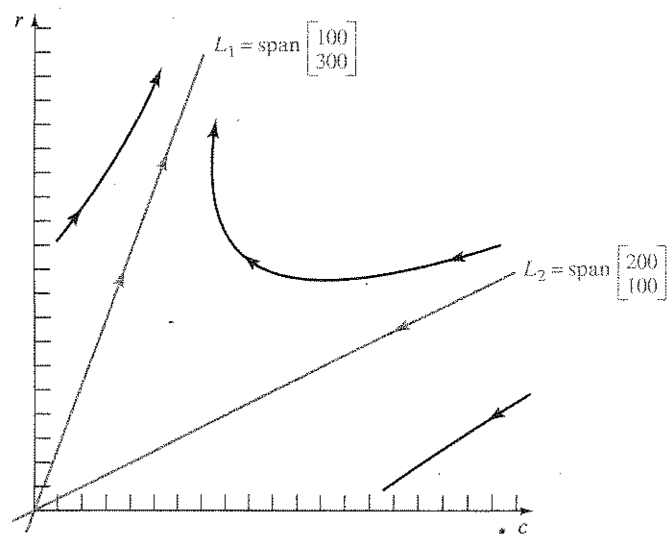


Figure 5

We see that the two populations will prosper over the long term if the ratio r_0/c_0 of the initial populations exceeds $1/2$; otherwise, both populations will die out.

From a mathematical point of view, it is informative to sketch a phase portrait of this system in the whole c–r-plane, even though the trajectories outside the first quadrant are meaningless in terms of our population study. (See Figure 6.)

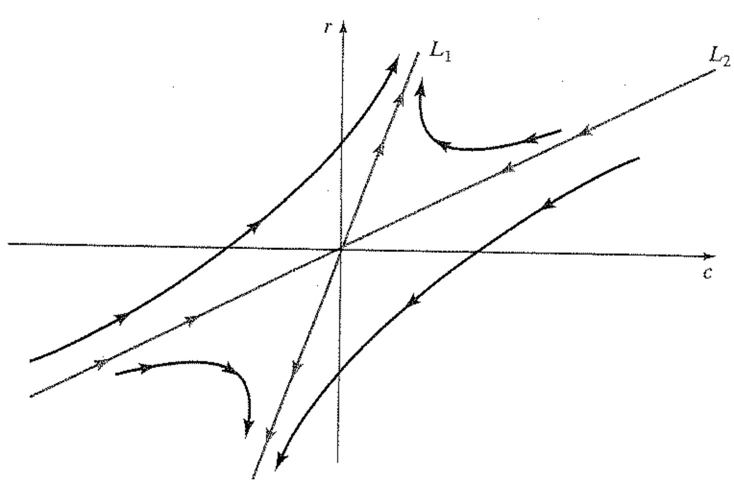


Figure 6