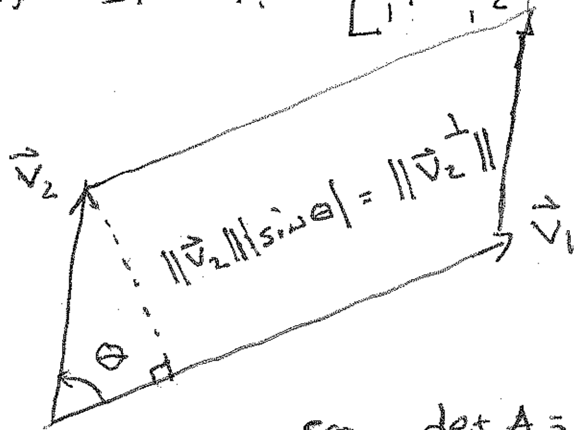
6.3: Geometric Interp of the Determinant.

(A) What we already know

(1) If $A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$, then $\det A$ = area of the parallelogram spanned by $\vec{v}_1$ & $\vec{v}_2$.



so $\det A = \|\vec{v}_1\|\,\|\vec{v}_2^{\perp}\|$.

This reminds us of our QR Factorization.

- as a prelim to this... if A is a square matrix and $A^TA = I$, then A is orthogonal. The converse is also true.

$$\Rightarrow \det(A^TA) = \det(A^T)\det(A) = (\det A)^2 = 1$$

for ~~orthotho~~ gonal matrices.

$$\Rightarrow \det(A) = \pm 1$$

- **Dfn**: An orthogonal $n\times n$ matrix A w/ $\det A = 1$ is called a rotation matrix.

- If $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ is invertible then...

$$A = QR \quad \text{and} \quad |\det A| = |\det(QR)|$$
$$= |\det(Q)||\det(R)|$$
$$= |\det(R)|$$
$$= \|\vec{v}_1\|\,\|\vec{v}_2^{\perp}\| \cdot \ldots \cdot \|\vec{v}_n^{\perp}\|$$

(2) We now have ~~easy~~ two ways to find the volume of the parallelpiped spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$

(i) $V = |\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)|$

(ii) if $A = QR = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$ then

$$V = \|\vec{v}_1\| \, \|\vec{v}_2^{\perp}\| \, \|\vec{v}_3^{\perp}\|$$

<u>Q</u>: what if $A_{n \times m}$ isn't square? (w/ LI cols)

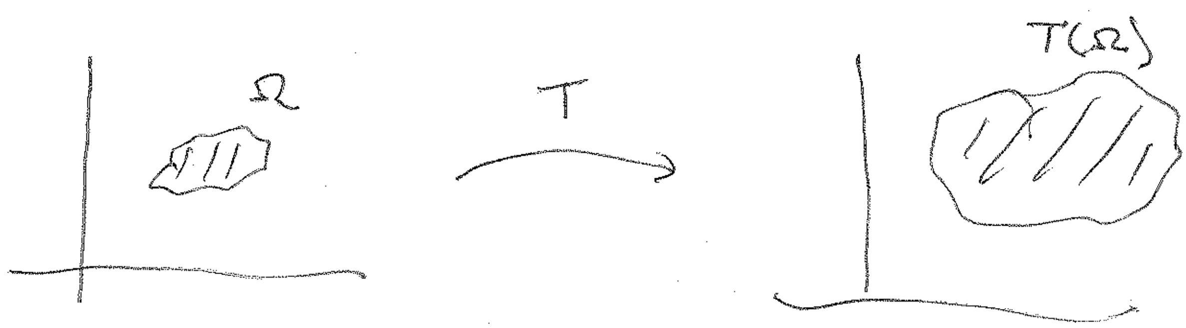
$A = QR$ and the square

$$A^T A = R^T Q^T Q R = R^T R$$

$$\Rightarrow \det(A^T A) = \det(R^T R) = (\det R)^2 = (\|v_1\| \|v_2^{\perp}\| \cdots \|v_m^{\perp}\|)^2$$

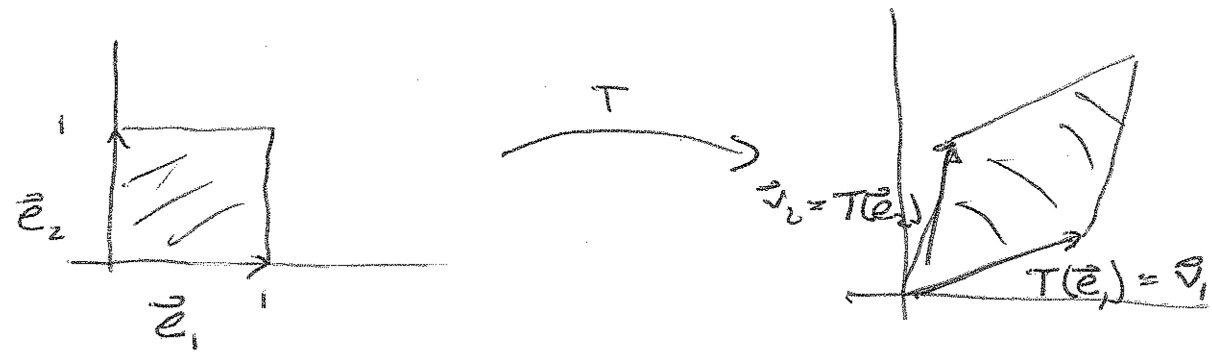
<u>Thm</u>: Consider the vecs $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$. The m-volume of the m-parallelepiped defined by $\vec{v}_1, \dots, \vec{v}_m$ is $\sqrt{\det(A^T A)}$ where A is the $n \times m$ matrix w/ cols $\vec{v}_1, \dots, \vec{v}_m$.

(B) The determinant is the expansion factor



$$\text{expansion factor} = \frac{\text{area}(T(\Omega))}{\text{area}(\Omega)}$$

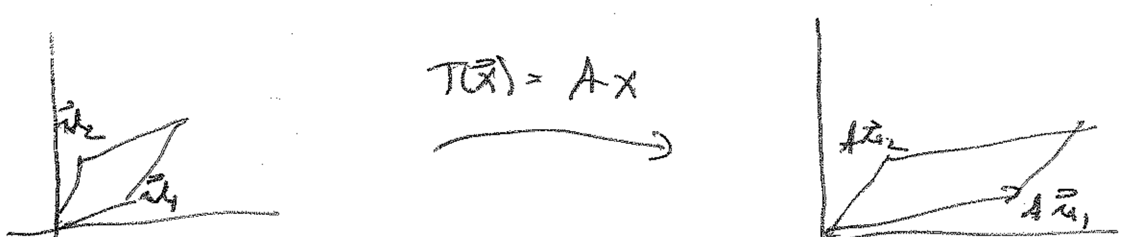
for a linear trans...



$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$.

so $\text{expansion factor} = \dfrac{|\det A|}{1} = \det A$

If we don't begin w/ the unit square.



$$\text{expansion factor} = \frac{\det\begin{bmatrix} A\vec{u}_1 & A\vec{u}_2 \end{bmatrix}}{\det\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}}$$

$$= \frac{\det A \det\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}}{\det\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}}$$

$$= \det A$$

This explains why $\det A^{-1} = \dfrac{1}{\det A}$.