5.3: Orthogonal Transformations

Dfn: A L.T. $T: \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the length of vectors.

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \text{for all } \vec{x} \in \mathbb{R}^n$$

The Pythagorean Thm for vectors.

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 \quad \text{iff} \quad \vec{x} \perp \vec{y}$$

Thm: Consider an orthogonal transformation $T: \mathbb{R}^n \mapsto \mathbb{R}^n$. If $\vec{v}, \vec{w} \in \mathbb{R}^n$ and $\vec{v} \perp \vec{w}$, then $T(\vec{v}) \perp T(\vec{w})$.

□ proof.

consider
$$\begin{aligned}\|T(\vec{v}) + T(\vec{w})\|^2 &= \|T(\vec{v}+\vec{w})\|^2 \\ &= \|(\vec{v}+\vec{w})\|^2 \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 \\ &= \|T(\vec{v})\|^2 + \|T(\vec{w})\|^2\end{aligned}$$

Therefore $T(\vec{v}) \perp T(\vec{w})$

Q.E.D.

This tells us that angles are preserved under orthogonal lin. trans.

Caution: A matrix w/ orthogonal cols. is not necessarily an orthogonal matrix.

ex: $M = \begin{bmatrix} 5 & -7 \\ 7 & 5 \end{bmatrix}$

Thm: A lin. trans. $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ form an orthonormal basis for $\mathbb{R}^n$.

Thm: An $n \times n$ matrix A is orthogonal iff its cols form an orthonormal basis of $\mathbb{R}^n$.

Q: How do you show a matrix is orthogonal?

Thm: Consider an $n \times n$ matrix A. A is orthogonal iff $A^T A = I_n$. (or $A^{-1} = A^T$)

☐ proof.

suppose $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

and $A^T = \begin{bmatrix} — & \vec{v}_1 & — \\ & \vdots & \\ — & \vec{v}_n & — \end{bmatrix}$

so $A^T A = \begin{bmatrix} — & \vec{v}_1 & — \\ & \vdots & \\ — & \vec{v}_n & — \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$

$$= \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vdots & & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \cdots & \vec{v}_n \cdot \vec{v}_n \end{bmatrix}$$

($\Rightarrow$) Assume A is orthogonal.

$\Rightarrow$ the cols are an orthonormal basis for $\mathbb{R}^n$.

$\Rightarrow$ $A^T A = I_n$ since $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & i=j \\ 0, & \text{else} \end{cases}$

($\Leftarrow$) Assume $A^T A = I_n$

$\Rightarrow$ $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & i=j \\ 0, & \text{else} \end{cases}$

$\Rightarrow$ cols of A are an orthonormal basis for $\mathbb{R}^n$

$\Rightarrow$ A is orthogonal.

Hence our claim is proved. ∎

Last Thm: Derivation

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

$$= \underbrace{(\vec{u}_1^T \vec{x})}_{\text{scalars}} \vec{u}_1 + \dots + \underbrace{(\vec{u}_m^T \vec{x})}_{\text{scalars}} \vec{u}_m$$

$$= \vec{u}_1 \vec{u}_1^T \vec{x} + \dots + \vec{u}_m \vec{u}_m^T \vec{x}$$

$$= (\vec{u}_1 \vec{u}_1^T + \dots + \vec{u}_m \vec{u}_m^T)\vec{x}$$

$$= \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} - & \vec{u}_1 & - \\ & \vdots & \\ - & \vec{u}_m & - \end{bmatrix} \vec{x}$$

$$= Q Q^T \vec{x}.$$

check the ij entry.

$(\vec{u}_1)_i (\vec{u}_1)_j + \dots$

$(\vec{u}_m)_i (\vec{u}_m)_j$

Hence we have that for a subspace V of $\mathbb{R}^n$ w/orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$. The matrix of the orthogonal projection onto V is

$$QQ^T \quad \text{where} \quad Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$$

Note: Q is $n \times m$ (not square).

Note: $QQ^T \neq Q^TQ$

<u>Orthogonal matrices</u>

Consider an $n \times n$ matrix A. Then the following are equivalent.

(1) A is an orthogonal matrix

(2) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$

(3) The cols of A form an orthonormal basis of $\mathbb{R}^n$.

(4) $A^TA = I_n$

(5) $A^{-1} = A^T$

The Transpose (3 properties)

(1) $(AB)^T = B^TA^T$

(2) $(A^T)^{-1} = (A^{-1})^T$

(3) $\text{rank}(A) = \text{rank}(A^T)$

Thm: The product AB of two orthogonal $n \times n$ matrices A & B is orthogonal.

Thm: The inverse A^{-1} of an orthogonal $n \times n$ matrix A is orthogonal.

Dfn: A is symmetric if $A = A^T$

Dfn: A is skewsymmetric if $A^T = -A$.

connect the dot product & matrix product.