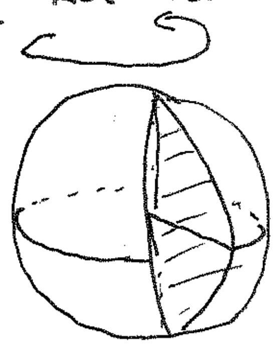
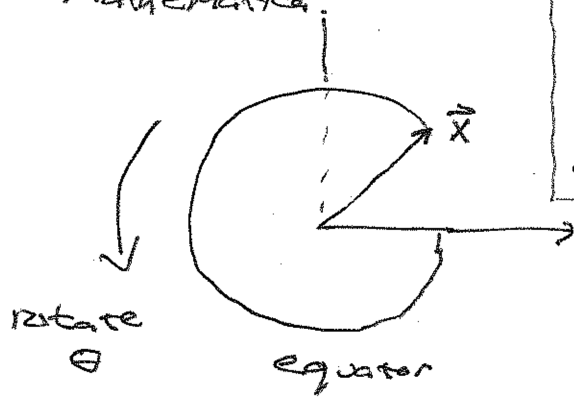
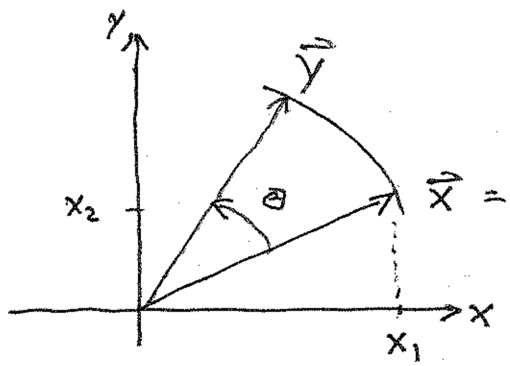
2.1: Linear Transformations

Ex1: Rotation using Mathematica.





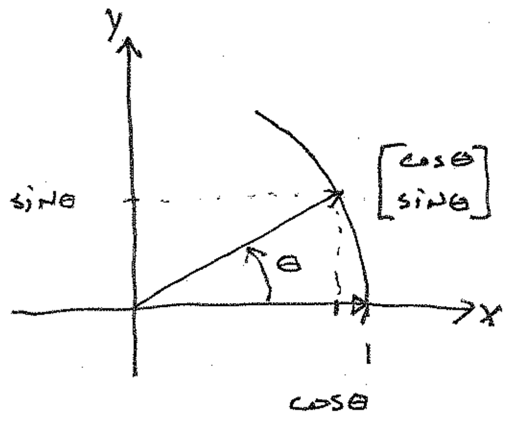
NOTE: The text introduces this w/ a delightful coast guard encryption example

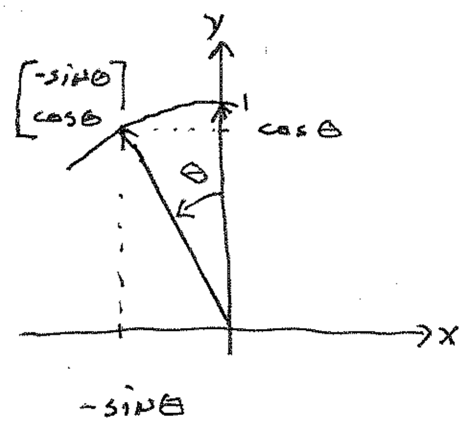


$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \overset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + x_2 \overset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Goal: Find $T(\vec{x})$ .. the fct that rotates $\vec{x}$ by $\vec{\theta}$ c.c.w.

Let's rotate $\vec{e}_1$ & $\vec{e}_2$ c.c.w. by the angle θ.





$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \longrightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \longrightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= A\vec{x}$$

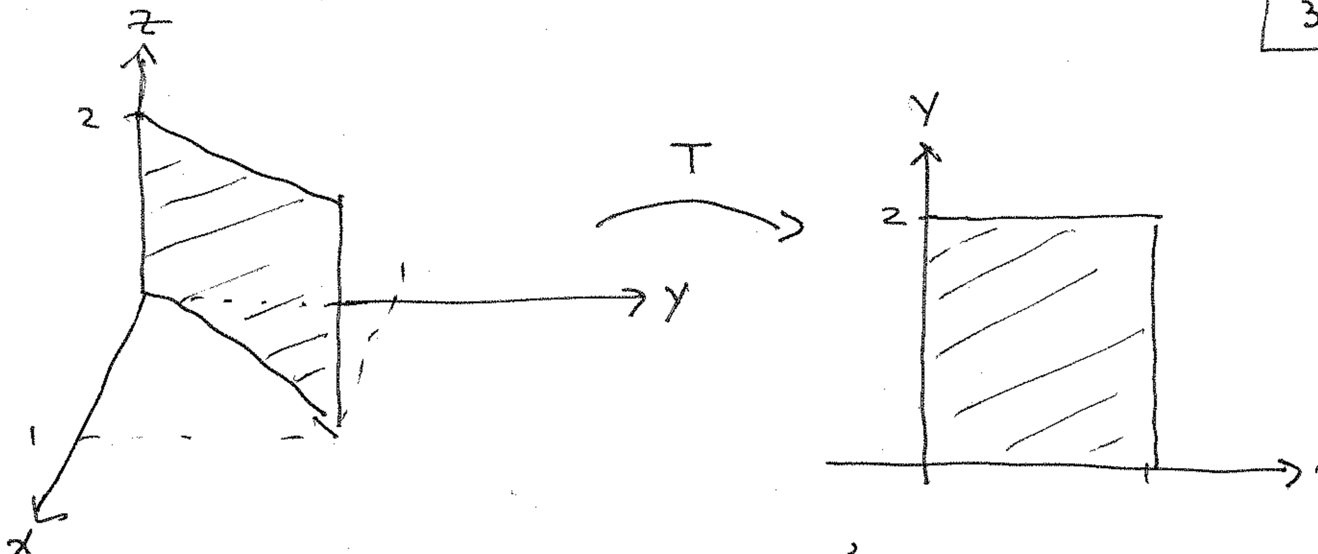
This rotation fct T is an example of a linear transformation.

<u>Dfn</u>: A fct $T: \mathbb{R}^m$ to $\mathbb{R}^n$ is called a <u>linear transformation</u> if $\exists\ A_{n\times m}$ s.t. $T(\vec{x}) = A\vec{x}$ $\forall\ \vec{x} \in \mathbb{R}^m$.

<u>Ex 2</u>: Consider the vectors $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right.$

under the transformation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



projects an image in $\mathbb{R}^3$ onto the plane.

Some transforms can be undone (ex1) and others can't (ex2). That is, the inverse of a linear transformation doesn't always exist.

The identity matrix/transformation.

The standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$

<u>Thm</u>: If T is a linear transformation, then the corresponding transformation matrix is.

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_m) \\ | & & | \end{bmatrix}_{n \times m}$$

To see this: If $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$, then $A\vec{e}_i = \vec{v}_i$ for $i = 1, \ldots, m$

recall $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

and $A(k\vec{v}) = kA\vec{v}$.

<u>Thm:</u> A transformation $T: \mathbb{R}^m \mapsto \mathbb{R}^n$ is linear iff

(a) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

(b) $T(k\vec{v}) = kT(\vec{v})$.

$\square$ <u>proof</u>.

($\Rightarrow$) for (b) (assume T is linear).

$T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v})$.

(a) is nearly identical.

($\Leftarrow$) (assume properties (a) & (b)).

consider $T(\vec{x})$ where $\vec{x} \in \mathbb{R}^m$.

$$
\begin{aligned}
T(\vec{x}) &= T(x_1\vec{e}_1 + \dots + x_m\vec{e}_m) \\
&= T(x_1\vec{e}_1) + \dots + T(x_m\vec{e}_m) \\
&= x_1T(\vec{e}_1) + \dots + x_mT(\vec{e}_m) \\
&= \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_m) \\ | & & | \end{bmatrix} \vec{x} \\
&= A\vec{x}. \quad \blacksquare
\end{aligned}
$$

What happens to $\vec{x}$ under the linear transformation T if $T(\vec{v}_1) = 2\vec{v}_1$ & $T(\vec{v}_2) = \frac{4}{3}\vec{v}_2$?