1.3: On the Solutions of linear Systems; matrix algebra.

I Prelim.

RREF: After Gauss - Jordan Elimination, we say a matrix is in reduced row echelon form. or RREF.

so given matrix A, we can find $\text{rref}(A)$.

Rank: We define the rank of matrix A, or $\text{rank}(A)$, as the number of leading ones in $\text{rref}(A)$.

read carefully thru ex 1-5 and thms 1.3.3 & 1.3.4 on your own.

II matrix algebra.

adding matrices (by element)

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{bmatrix}_{n\times m} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}_{n\times m} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}_{n\times m}$$

multiplying a matrix by a scalar. (by element/entry)

$$k\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$$

matrix multiplication (row by column).

ex1: Find $\underset{3\times 2}{A} \cdot \underset{2\times 4}{B} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \\ 1 & 2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 4 & 5 & 2 & 3 \end{bmatrix}_{2\times 4}$

$\vec{w}_1, \vec{w}_2, \vec{w}_3$; $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

same

dim of product.

we are using $\vec{w}_i$'s for rows and $\vec{v}_j$'s for cols.

Notice our old friend the dot product ...

ex2: $\vec{w}_1 \cdot \vec{v}_1 = \begin{bmatrix} 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 26$

(the dot product of a row & col vector is a scalar).

more generally.

$$\underset{n\times m}{A} \cdot \underset{m\times p}{B} = \begin{bmatrix} - \vec{w}_1 - \\ \vdots \\ - \vec{w}_n - \end{bmatrix} \cdot \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \vec{w}_1 \cdot \vec{v}_1 & \cdots & \vec{w}_1 \cdot \vec{v}_p \\ \vdots & & \vdots \\ \vec{w}_n \cdot \vec{v}_1 & \cdots & \vec{w}_n \cdot \vec{v}_p \end{bmatrix}_{n\times p}$$

Two ways to write $A\vec{x}$

and if

like $\vec{v}_1$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \text{ then } A\vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}_{n\times 1}$$

We want another expression for $A\vec{x}$

ex 3: $A\vec{x} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \\ 1 & 2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2\times 1}$

$$= \begin{bmatrix} 2(1) + 6(4) \\ 0(1) + 4(4) \\ 1(1) + 2(4) \end{bmatrix}$$

$$= 1\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

and $A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \ldots + x_m\vec{v}_m$ ②

Dfn: A vector $\vec{b} \in \mathbb{R}^n$ is called a linear combination of $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ if there exist scalars $x_1, \ldots, x_m$ s.t. $\vec{b} = x_1\vec{v}_1 + \ldots + x_m\vec{v}_m$.

So $A\vec{x}$ is a linear combo of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$

This will be used extensively in the theoretical areas. We can see...

$A\vec{x} = x_1\vec{v}_1 + \ldots + x_m\vec{v}_m$ (you can break apart $A\vec{x}$)

and

$x_1\vec{v}_1 + \ldots + x_n\vec{v}_m = A\vec{x}$ (you can combine a lin. combo into $A\vec{x}$).

Two rules for $A\vec{x}$: If A is an $n \times m$ matrix and $\vec{x}, \vec{y} \in \mathbb{R}^n$, and k is a scalar

(a) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

(b) $A(k\vec{x}) = kA\vec{x}$

□ proof of (b).

$$A(k\vec{x}) = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} kx_1 \\ \vdots \\ kx_m \end{bmatrix} = \begin{bmatrix} a_{11}kx_1 + \dots + a_{1m}kx_m \\ \vdots \\ a_{n1}kx_1 + \dots + a_{nm}kx_m \end{bmatrix}$$

$$= k \begin{bmatrix} a_{11}x_1 + \dots + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + \dots + a_{nm}x_m \end{bmatrix}$$

$$= k \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= kA\vec{x} \quad \blacksquare$$

□ proof of (a) ... alternative prf is clearer.

$$A(k\vec{x}) = A(k \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix})$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} kx_1 \\ \vdots \\ kx_m \end{bmatrix}$$

$$= kx_1\vec{v}_1 + \dots + kx_m\vec{v}_m$$

$$= k(x_1\vec{v}_1 + \dots + x_m\vec{v}_m)$$

$$= k \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = kA\vec{x} \quad \blacksquare$$

1.3
5/5

Linear System: $\left| \begin{array}{l} x_1 + 2x_2 = 8 \\ 3x_1 - x_2 = 3 \end{array} \right|$

w/ augmented matrix $[A \mid b]$ $\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -1 & 3 \end{array}\right]$

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$