## True or False

**Ch 2.TF.1** T, by Theorem 2.2.4.

**Ch 2.TF.2** T, by Theorem 2.4.6.

**Ch 2.TF.3** T; The matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

**Ch 2.TF.4** F; The columns of a rotation matrix are unit vectors; see Theorem 2.2.3.

**Ch 2.TF.5** T, by Theorem 2.4.3.

**Ch 2.TF.6** T; Let $A = B$ in Theorem 2.4.7.

**Ch 2.TF.7** F, by Theorem 2.3.3.

**Ch 2.TF.8** T, by Theorem 2.4.8.

**Ch 2.TF.9** F; Matrix $AB$ will be $3 \times 5$, by Definition 2.3.1b.

**Ch 2.TF.10** F; Note that $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A linear transformation transforms $\vec{0}$ into $\vec{0}$.

**Ch 2.TF.11** T; The equation $\det(A) = k^2 - 6k + 10 = 0$ has no real solution.

**Ch 2.TF.12** T; The matrix fails to be invertible for $k = 5$ and $k = -1$, since the determinant $\det A = k^2 - 4k - 5 = (k-5)(k+1)$ is 0 for these values of $k$.

**Ch 2.TF.13** F; Note that $\det(A) = (k-2)^2 + 9$ is always positive, so that $A$ is invertible for all values of $k$.

**Ch 2.TF.14** F We can show by induction on $m$ that the matrix $A^m$ is of the form $A^m = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$ for all $m$, so that $A^m$ fails to be positive. Indeed, $A^{m+1} = A^m A = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$.

**Ch 2.TF.15** F; Consider $A = I_2$ (or any other invertible $2 \times 2$ matrix).

**Ch 2.TF.16** T; Note that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$ is the unique solution.

**Ch 2.TF.17** F, by Theorem 2.4.9. Note that the determinant is 0.

**Ch 2.TF.18** T, by Theorem 2.4.3.

**Ch 2.TF.19** T; The shear matrix $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ works.

**Ch 2.TF.20** T; Simplify to see that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ -12x \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -12 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

**Ch 2.TF.21** F; If matrix $A$ has two identical rows, then so does $AB$, for any matrix $B$. Thus $AB$ cannot be $I_n$, so that $A$ fails to be invertible.

**Ch 2.TF.22** T, by Theorem 2.4.8. Note that $A^{-1} = A$ in this case.

**Ch 2.TF.23** F; For any $2 \times 2$ matrix $A$, the two columns of $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ will be identical.

**Ch 2.TF.24** T; One solution is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

**Ch 2.TF.25** F; A reflection matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Here, $a^2 + b^2 = 1 + 1 = 2$.

**Ch 2.TF.26** T Let $B$ be the matrix whose columns are all $\vec{x}_{equ}$, the equilibrium vector of $A$.

**Ch 2.TF.27** T; The product is $\det(A)I_2$.

**Ch 2.TF.28** T; Writing an upper triangular matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and solving the equation $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we find that $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where $b$ is any nonzero constant.

**Ch 2.TF.29** T; Note that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a rotation through $\pi/2$. Thus $n = 4$ (or any multiple of 4) works.

**Ch 2.TF.30** F; If a matrix $A$ is invertible, then so is $A^{-1}$. But $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ fails to be invertible.

**Ch 2.TF.31** T For example, $A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

**Ch 2.TF.32** F Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, with $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

**Ch 2.TF.33** F; Consider matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, for example.

Ch 2.TF.**34** T; Apply Theorem 2.4.8 to the equation $(A^2)^{-1}AA = I_n$, with $B = (A^2)^{-1}A$.

Ch 2.TF.**35** F; Consider the matrix $A$ that represents a rotation through the angle $2\pi/17$.

Ch 2.TF.**36** F; Consider the reflection matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Ch 2.TF.**37** T; We have $(5A)^{-1} = \frac{1}{5}A^{-1}$.

Ch 2.TF.**38** T; The equation $A\vec{e}_i = B\vec{e}_i$ means that the $i$th columns of $A$ and $B$ are identical. This observation applies to all the columns.

Ch 2.TF.**39** T; Note that $A^2B = AAB = ABA = BAA = BA^2$.

Ch 2.TF.**40** T; Multiply both sides of the equation $A^2 = A$ with $A^{-1}$.

Ch 2.TF.**41** T See Exercise 2.3.75

Ch 2.TF.**42** F Consider $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$, with $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.

Ch 2.TF.**43** F; Consider $A = I_2$ and $B = -I_2$.

Ch 2.TF.**44** T; Since $A\vec{x}$ is on the line onto which we project, the vector $A\vec{x}$ remains unchanged when we project again: $A(A\vec{x}) = A\vec{x}$, or $A^2\vec{x} = A\vec{x}$, for all $\vec{x}$. Thus $A^2 = A$.

Ch 2.TF.**45** T; If you reflect twice in a row (about the same line), you will get the original vector back: $A(A\vec{x}) = \vec{x}$, or, $A^2\vec{x} = \vec{x} = I_2\vec{x}$. Thus $A^2 = I_2$ and $A^{-1} = A$.

Ch 2.TF.**46** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

Ch 2.TF.**47** T; Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.

Ch 2.TF.**48** F; By Theorem 1.3.3, there is a nonzero vector $\vec{x}$ such that $B\vec{x} = \vec{0}$, so that $AB\vec{x} = \vec{0}$ as well. But $I_3\vec{x} = \vec{x} \neq \vec{0}$, so that $AB \neq I_3$.

Ch 2.TF.**49** T; We can rewrite the given equation as $A^2 + 3A = -4I_3$ and $-\frac{1}{4}(A + 3I_3)A = I_3$. By Theorem 2.4.8, the matrix $A$ is invertible, with $A^{-1} = -\frac{1}{4}(A + 3I_3)$.

Ch 2.TF.**50** T; Note that $(I_n + A)(I_n - A) = I_n^2 - A^2 = I_n$, so that $(I_n + A)^{-1} = I_n - A$.

Ch 2.TF.**51** F; $A$ and $C$ can be two matrices which fail to commute, and $B$ could be $I_n$, which commutes with anything.

**Ch 2.TF.52** F; Consider $T(\vec{x}) = 2\vec{x}$, $\vec{v} = \vec{e}_1$, and $\vec{w} = \vec{e}_2$.

**Ch 2.TF.53** F; Since there are only eight entries that are not 1, there will be at least two rows that contain only ones. Having two identical rows, the matrix fails to be invertible.

**Ch 2.TF.54** F; Let $A = B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, for example.

**Ch 2.TF.55** F; We will show that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S$ fails to be diagonal, for an arbitrary invertible matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$. Since $c$ and $d$ cannot both be zero (as $S$ must be invertible), at least one of the off-diagonal entries ($-c^2$ and $d^2$) is nonzero, proving the claim.

**Ch 2.TF.56** T; Consider an $\vec{x}$ such that $A^2\vec{x} = \vec{b}$, and let $\vec{x}_0 = A\vec{x}$. Then $A\vec{x}_0 = A(A\vec{x}) = A^2\vec{x} = \vec{b}$, as required.

**Ch 2.TF.57** T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $A^{-1} = -A$, or $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. This holds if $ad - bc = 1$ and $d = -a$. These equations have many solutions: for example, $a = d = 0, b = 1, c = -1$. More generally, we can choose an arbitrary $a$ and an arbitrary nonzero $b$. Then, $d = -a$ and $c = -\frac{1+a^2}{b}$.

**Ch 2.TF.58** F; Consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We make an attempt to solve the equation $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now the equation $b(a+d) = 0$ implies that $b = 0$ or $d = -a$.

If $b = 0$, then the equation $d^2 + bc = -1$ cannot be solved.

If $d = -a$, then the two diagonal entries of $A^2$, $a^2+bc$ and $d^2+bc$, will be equal, so that the equations $a^2+bc = 1$ and $d^2+bc = -1$ cannot be solved simultaneously.

In summary, the equation $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ cannot be solved.

**Ch 2.TF.59** T; Recall from Definition 2.2.1 that a projection matrix has the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector. Thus, $a^2+b^2+c^2+d^2 = u_1^4 + (u_1u_2)^2 + (u_1u_2)^2 + u_2^4 = u_1^4 + 2(u_1u_2)^2 + u_2^4 = (u_1^2+u_2^2)^2 = 1^2 = 1$.

**Ch 2.TF.60** T; We observe that the systems $AB\vec{x} = 0$ and $B\vec{x} = 0$ have the same solutions (multiply with $A^{-1}$ and $A$, respectively, to obtain one system from the other). Then, by True or False Exercise 45 in Chapter 1, rref$(AB)$ =rref$(B)$.

**Ch 2.TF.61** T For example, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with $A^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for even $m$ and $A^m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for odd $m$.

**Ch 2.TF.62** T We need to show that the system $A\vec{x} = \vec{x}$ or $(A - I_n)\vec{x} = \vec{0}$ has a nonzero solution $\vec{x}$. This amounts to showing that rank$(A - I_n) < n$, or, equivalently, that rref$(A - I_n)$ has a row of zeros. By definition of a transition matrix, the sum of all the row vectors of $A$ is $\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$, so that the sum of all the row vectors of $A - I_n$ is the zero row vector. If we add rows $I$ through $(n-1)$ to the last row of $A - I_n$, we generate a row of zeros as required.