9:26

Test 1
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Math 153

No work = no credit

No Symbolic Calculators

Name: Key

I myself, a professional mathematician, on re-reading my own work find it strains my mental powers to recall to mind from the figures the meanings of the demonstrations, meanings which I myself originally put into the figures and the text from my mind. But when I attempt to remedy the obscurity of the material by putting in extra words, I see myself falling into the opposite fault of becoming chatty in something mathematical.

Johannes Kepler (1597 - 1630)
German astronomer

Warm-ups (1 pt each): $\vec{i}\cdot\vec{i} =$ 1 $\qquad \vec{i}\times\vec{i} =$ $\vec{0}$ $\qquad \vec{i}-\vec{i} =$ $\vec{0}$

1.) (1 pt) Based upon the quote above, how did easily did Kepler understand his earlier work? Answer using complete English sentences.

He had a hard time understanding.

2.) Consider points $A(2,4,5)$, $B(1,5,7)$, and $C(-1,6,8)$.

a.) Find the equation of the line that includes *A* and *B*.
b.) Find the angle between AB and AC. Express your answer in radians. to 4 places.
c.) Find the equation of the plane that includes the three points.

(a) $\vec{AB} = \langle -1, 1, 2\rangle$ 2pts $\qquad 2-x = y-4 = \frac{z-5}{2}$

$\vec{r}(t) = \langle 2-t, 4+t, 5+2t\rangle$

(b) $\vec{AC} = \langle -3, 2, 3\rangle$

$\underbrace{\vec{AB}\cdot\vec{AC}}_{11} = \underbrace{|\vec{AB}|}_{\sqrt{6}}\,\underbrace{|\vec{AC}|}_{\sqrt{22}}\cos\theta$

$\theta = \arccos\left(\frac{11}{\sqrt{132}}\right) \approx 0.2928$

(c) $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = \langle -1, -3, 1\rangle$

$-(x-2) + -3(y-4) + 1(z-5) = 0$

$-x - 3y + z = -9$

3.) Consider the line $\vec{r}(t) = \langle 2t, 1+2t, 2t \rangle$ and the point $A(2,1,-1)$.

a.) Find the plane that includes point *A* that is perpendicular to the line.

b.) Find the point B where the line intersects the plane found in (a.).

c.) Find the distance from point *A* to the line.

(a) $\vec{N} = \langle 2, 2, 2 \rangle$

plane: $2(x-2) + 2(y-1) + 2(z+1) = 0$

$2x + 2z + 2z = 4$

(b) pt: $2(2t) + 2(1+2t) + 2(2t) = 4$

$\Rightarrow 12t + 2 = 4$

$\Rightarrow t = \frac{1}{6}$

$\Rightarrow$ pt $B\left(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right)$

(c) $D(A,B) = \sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$

$= \frac{\sqrt{42}}{3}$

4.) Consider the parametric curve $x(t) = e^t$ and $y(t) = 2e^t$ for $t \geq 0$.

a.) Compute the length of the curve on $0 \leq t \leq 1$

b.) Compute the length of the curve on $1 \leq t \leq 2$

c.) Convert this parametric equation into an equation of the form $y = f(x)$ and compute the length of the graph of $f(x)$ on $1 \leq x \leq 2$.

d.) Is the answer in part (c.) the same as the answer in part (b.)? How do you explain this result?

(a) Line $y = 2x$ from $(1,2)$ to $(e, 2e)$

$$D = \sqrt{(e-1)^2 + (2e-2)^2}$$

$$= \sqrt{5(e-1)^2}$$

$$= \sqrt{5}\,(e-1).$$

(b) from $(e, 2e)$ to $(e^2, 2e^2)$

$$D = \sqrt{(e^2 - e)^2 + (2e^2 - 2e)^2}$$

$$= \sqrt{5}\,(e^2 - e)$$

(c) $y = 2x$

from $(1,2)$ to $(2,4)$

$$D = \sqrt{1+4} = \sqrt{5}$$

(d) No. The line is parametrized differently.

5.) Consider the limaçon $r = 2\cos(\theta)+1$. $\quad 2\cos\theta + 1 = 0 \Rightarrow \theta = \arccos\left(-\frac{1}{2}\right)$

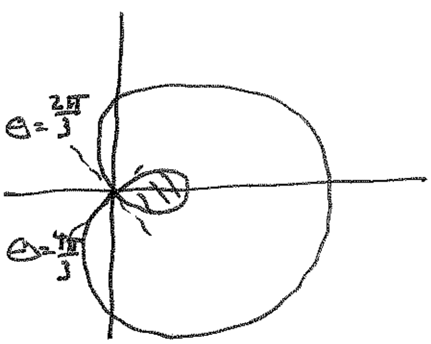
a.) Find the first quadrant point with a horizontal tangent.

b.) Find the second quadrant point where the tangent is vertical.

c.) Set up (do not solve) an integral that represents the area inside the smaller loop of the limaçon.

$y = r\sin\theta \qquad x = r\cos\theta$

(a)
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$



$$= \frac{-2\sin\theta\cdot\sin\theta + (2\cos\theta+1)\cos\theta}{-2\sin\theta\cdot\cos\theta \mp (2\cos\theta+1)\sin\theta}$$

$$= \frac{-2\sin^2\theta + 2\cos^2\theta + \cos\theta}{-2\sin\cancel{\theta}\cos\theta \mp 2\sin\cancel{\theta}\cos\theta \mp \sin\theta}$$

$$= \frac{2\cos^2\theta + \cos\theta + -2(1-\cos^2\theta)}{-\sin\theta(1+4\cos\theta)}$$

$$\frac{dy}{dx} = 0 \Rightarrow 4\cos^2\theta + \cos\theta - 2 = 0$$

$$\cos\theta = \frac{-1 \pm \sqrt{1-4(4)(-2)}}{2(4)}$$

$$\theta = \arccos\left(\frac{-1\pm\sqrt{33}}{8}\right)$$

$$\approx 0.9359$$

(b) $\frac{dy}{dx}$ undefined. $\quad 1+4\cos\theta = 0 \Rightarrow \theta = \arccos\left(-\frac{1}{4}\right)$

≈ 1.8235

(c) $$A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+2\cos\theta)^2\, d\theta$$

$(r,\theta) = \left(\frac{1}{2}, \cos^{-1}\left(-\frac{1}{4}\right)\right)$ point in polar coordinates.

6.) A point moves along the helix with position vector given by $\vec{r}(t) = \langle 6\sin(2t),\, 5t,\, 6\cos(2t) \rangle$.

a.) Find $\vec{v}$ and $\vec{a}$.

~~b.) Write $\vec{a}$ as a combination of $\vec{r}$ and $\vec{v}$.~~

c.) Find the unit tangent vector $\vec{T}(t)$ for $t \neq 0$

d.) Find the unit normal vector $\vec{N}(t)$ for $t \neq 0$

e.) Find the curvature of $\vec{r}$

f.) Find the tangential and normal components of the acceleration vector.

g.) Show that $\vec{a} = a_t\vec{T} + a_N\vec{N}$

(a) $\vec{r}' = \vec{v} = \langle 12\cos 2t,\ 5,\ -12\sin 2t \rangle$

$\vec{r}'' = \vec{a} = \langle -24\sin 2t,\ 0,\ -24\cos 2t \rangle$

(b.) $|\vec{r}'| = \sqrt{144\cos^2 2t + 25 + 144\sin^2 2t} = 13$

$\vec{T}(t) = \left\langle \frac{12}{13}\cos 2t,\ \frac{5}{13},\ \frac{-12}{13}\sin 2t \right\rangle$

(c.) $\vec{T}'(t) = \left\langle -\frac{24}{13}\sin 2t,\ 0,\ -\frac{24}{13}\cos 2t \right\rangle$

$|\vec{T}'(t)| = \sqrt{\left(\frac{24}{13}\right)^2\sin^2 2t + \left(\frac{24}{13}\right)^2\cos^2 2t} = \frac{24}{13}$

$\vec{N}(t) = \langle -\sin 2t,\ 0,\ -\cos 2t \rangle$

(d) $\cancel{k=}$ $\vec{r}' \times \vec{r}'' = \langle -120\cos 2t,\ 288,\ 120\sin 2t \rangle$

$k = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \dfrac{\sqrt{97344}}{13^3} = \dfrac{312}{13^3} = \dfrac{24}{13^2}$

(e) $a_T = 0$ and $a_N = \dfrac{\sqrt{24^2 \cdot 13^2}}{13} = 24$