# 12.4 : Cross Product

## Determinants

**Ex 1:** Is there a unique solution to the system
$$\begin{cases} x + 2y = c_1 \\ 3x + 4y = c_2 \end{cases}$$

To answer this, calculate the determinate
$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3).$$

If the determinate is not zero then a unique solution exists.

**Ex 2:** Is there a unique solution to the system
$$\begin{cases} x + 2y + 3z = c_1 \\ 4x + 5y + 6z = c_2 \\ 7x + 8y + 9z = c_3 \end{cases}$$

calculate the determinate

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35)$$
$$= -3 + 12 - 9$$
$$= 0$$

So, there is not a unique solution.

<u>Definition</u> If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the <u>cross product</u> is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2 v_3 - u_3 v_2, \ \ldots \rangle$$

<u>Ex 3</u>: If $\vec{u} = \langle 1, 2, 3 \rangle$ and $\vec{v} = \langle 4, 5, 6 \rangle$, find $\vec{u} \times \vec{v}$.

<u>Ex 4</u>: If $\vec{u} = \langle 1, -1, 2 \rangle$ and $\vec{v} = \langle -3, 5, -8 \rangle$, find $\vec{u} \times \vec{v}$.

<u>Ex 5</u>: Show $\vec{u} \times \vec{u} = \vec{0} \quad \forall \vec{u} \in V_3$

<u>Thm</u>: The vector $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

<u>Ex 6</u>: Show that $\vec{u} \times \vec{v}$ in (ex 3) is orthogonal to $\vec{u}$ and $\vec{v}$ (dot product). Could also find the angle between $\vec{u} \times \vec{v}$ & $\vec{u}$.

Geometrically, there are two directions that could be taken by a vector $\perp$ to the span of $\vec{u}$ & $\vec{v}$. $\vec{u} \times \vec{v}$ has direction fixed by the right-hand rule.

Now that we have the direction of $\vec{u} \times \vec{v}$, what is its magnitude?

<u>Thm:</u> If $\theta$ is the angle between $\vec{u}$ & $\vec{v}$ $(0 \leq \theta \leq \pi)$, then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

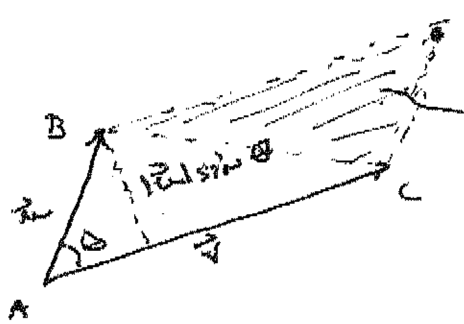
□ <u>proof.</u>

In the book, ... read it backwards.

<u>Corollary</u>: Two non-zero vectors $\vec{u}$ & $\vec{v}$ are parallel iff $\vec{u} \times \vec{v} = \vec{0}$.

<u>EX 7</u>: Find a vector $\perp$ to the plane thru $A(1,2,3)$, $B(4,5,6)$, and $C(7,8,9)$.

<u>Ex 8</u>: Find the area of the triangle w/ vertices $A(1,2,3)$, $B(4,5,6)$, and $C(7,8,9)$




Area of triangle is $\frac{1}{2}$ the area of the parallelogram. $= \dfrac{|\vec{u}||\vec{v}| \sin\theta}{2}$

OR, the parallelogram has area $|\vec{u} \times \vec{v}|$.

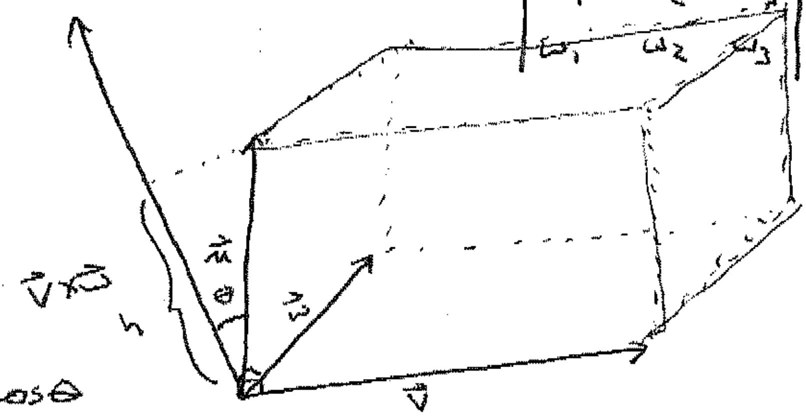
That is, the length of $\vec{u} \times \vec{v}$ equals the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

## Cross product properties

If $\vec{u}, \vec{v}, \vec{w} \in V_3$ and $c$ is a scalar

1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
5) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ scalar triple product.
6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Notice that $u \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$




$h = |\vec{u}| \cos\theta$

$$V = A \cdot h = |\vec{v} \times \vec{w}||\vec{u}| \cos\theta$$
$$= |\vec{u}||\vec{v} \times \vec{w}| \cos\theta$$
$$= |u \cdot (\vec{v} \times \vec{w})|$$

volume of the parallelopiped determined by $\vec{u}, \vec{v}$, and $\vec{w}$.

coplaner?