

12.3: The Dot Product

Defn: If $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$. This is called the dot product of \vec{u} and \vec{v} .

Ex1: Find $\langle 1, -1 \rangle \cdot \langle -2, 3 \rangle$

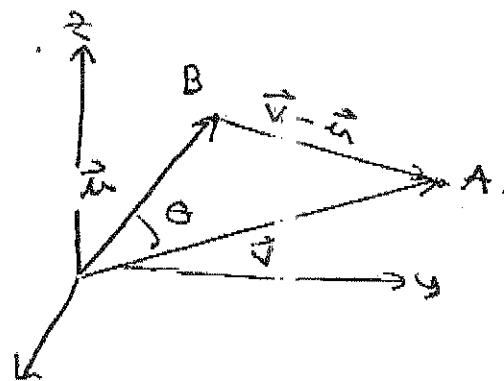
notice that the dot product gives a scalar result. Hence, it is sometimes called the scalar product.

Ex2: $(5\vec{i} - 8\vec{j} + 13\vec{k}) \cdot (+2\vec{i} + 3\vec{j} - 5\vec{k})$.

Properties of the dot product. $a, b, c \in \mathbb{V}_n$ and $c \in \mathbb{R}$

- 1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- 2) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- 3) $\vec{0} \cdot \vec{a} = 0$
- 4) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- 5) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

Geometrically, the dot product is related to the angle between vectors.



Thm: If θ is the angle between \vec{u} and \vec{v} , then $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos(\theta)$

proof: use the law of cosines.

~~$$|\vec{v} - \vec{u}|^2 = |\vec{v}|^2 + |\vec{u}|^2 - 2|\vec{v}||\vec{u}|\cos(\theta)$$~~

$$\Rightarrow |\vec{v}|^2 + |\vec{u}|^2 - 2|\vec{v}||\vec{u}|\cos(\theta) = |\vec{v} - \vec{u}|^2$$

$$= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u})$$

$$= |\vec{v}|^2 - 2\vec{v} \cdot \vec{u} + |\vec{u}|^2$$

$$\Rightarrow -2|\vec{v}||\vec{u}|\cos\theta = -2\vec{v} \cdot \vec{u}$$

$$\Rightarrow \vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}|\cos\theta //$$

EX3: If vectors \vec{u} and \vec{v} have lengths 7 and 5 w/ angle between of $\frac{\pi}{4}$, find $\vec{u} \cdot \vec{v}$.

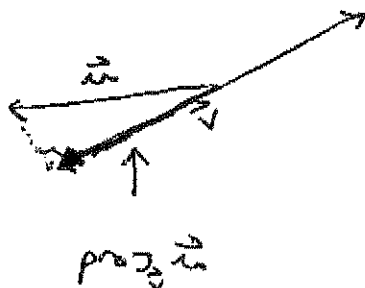
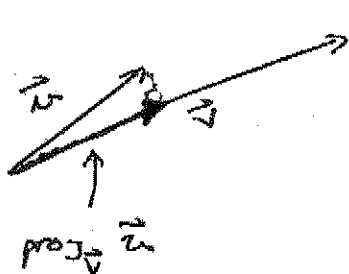
Corollary: If θ is the angle between vectors \vec{u} & \vec{v} , then $\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$.

EX4: Find the angle between $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$

* u & v are perpendicular (or orthogonal) iff $\vec{u} \cdot \vec{v} = 0$.

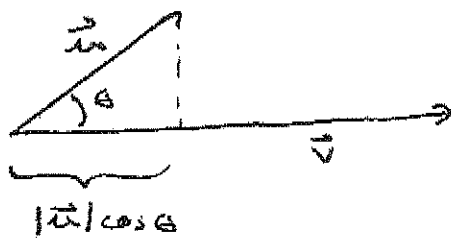
projections

In applications, we frequently project vectors onto each other. For example.



Let's find it...

We call the length of $\text{proj}_{\vec{v}} \vec{u} = \text{comp}_{\vec{v}} \vec{u}$



So, $\text{comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta$

Now consider $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$
 $= |\vec{v}| (|\vec{u}| \cos \theta)$

$\Rightarrow \text{comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

To find $\text{proj}_{\vec{v}} \vec{u}$, multiply the unit vector in the direction of \vec{v} by the length

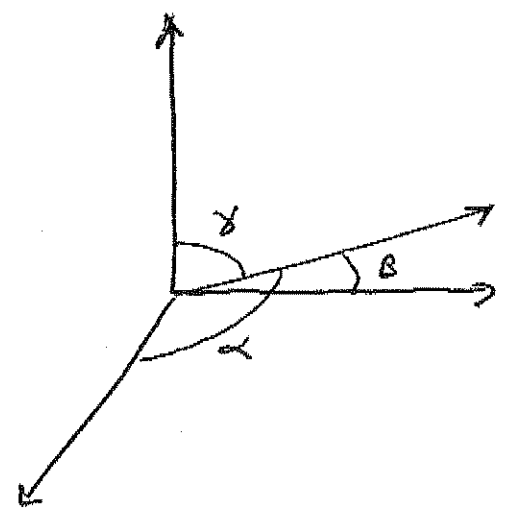
$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|} = \underbrace{\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right)}_{\text{scalar}} \vec{v}$

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Ex 5: Find the projection of $\langle 1, 2, 3 \rangle$ onto $\langle 4, 5, 6 \rangle$

Ex 6: Find the direction cosines by finding the angles between the vector and each axis

use that $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$



~~so $|\vec{u}| = \sqrt{1^2 + 2^2 + 3^2}$~~

so $\vec{u} = \frac{1}{|\vec{u}|} \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$.

Ex 7: Find the direction cosines of $\langle 1, 2, 3 \rangle$.