11.9: Representations of Fcts. as Power Series

Recall $f(u) = \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}$, $|u| < 1$.

Ex 1: Find a power series representation of $\frac{1}{1+x^2}$ and the I.O.C.

Ex 2: Find a power series representation of $\frac{1}{x+7}$ and the I.O.C.

Hint: $\frac{1}{x+7} = \frac{1}{7+x} = \frac{1}{7(1+x/7)} = \frac{1}{7(1-(-x/7))}$

Ex 3: same as #2, $\frac{x^8}{x+7}$.

Theorem: If the power series $\sum c_n(x-a)^n$ has R.O.C. $R>0$, then

i) $\frac{d}{dx}\left[\sum c_n(x-a)^n\right] = \sum \frac{d}{dx}\left[c_n(x-a)^n\right]$

ii) $\int \left[\sum c_n(x-a)^n\right]dx = \sum \int \left[c_n(x-a)^n dx\right]$

NOTE: Don't forget the constant of integration.
NOTE: The ROC stays constant

Ex 4: Express $\frac{2}{(1-x)^3}$ as a power series.

Notice that this looks a derivative of $\frac{1}{1-x}$.

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

and $$\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

so $$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\frac{1}{1-x}$$

$$= \frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)\, x^{n-2}$$

Ex 5: Find a power series for $\ln|2x-1|$

Since $\frac{d}{dx}\ln|2x-1| = \frac{2}{2x-1}$

we want $\int \frac{2}{2x-1}\,dx$

Ex 6: Find a power series rep for $f(x) = \tan^{-1}x$.

$\tan^{-1}x = \int \frac{dx}{1+x^2}$

side note in text on Leibniz formula for π.

Fibonacci Sequence: 1, 1, 2, 3, 5, 8, …

$$
\begin{array}{lcccccccc}
 & 1 & 1 & 2 & 3 & 5 & 8 & \ldots & \\
 & & 1 & 1 & 2 & 3 & 5 & \ldots & \\
\hline
\text{Add} & 1 & 2 & 3 & 5 & 8 & 13 & \ldots &
\end{array}
$$

just missing the 1st 1.

Let $F(x) = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + \ldots$

$$
\begin{array}{rl}
1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots & = x\,F(x) \\
1x^2 + 1x^3 + 2x^4 + 3x^5 + 5x^6 + \ldots & = x^2 F(x) \\
\hline
1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots &
\end{array}
$$

missing term.

$$\Rightarrow \quad F(x) - 1 = x\,F(x) + x^2 F(x)$$

$$\Rightarrow \quad F(x) = \frac{1}{1 - (x + x^2)}$$

$$= \sum_{n=0}^{\infty} (x + x^2)^n$$

This in turn can be simplified w/ the binomial series.