10.2: calculus w/parametric curves

Our goal is to explore vector fct which are related to parametric curves... An overview of the calculus of parametric curves is as follows.

If $x(t)$ and $y(t)$ are parametric functions for x and y, then...

1. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0.$$

Note: This is $\frac{dy}{dx}$ and not $\frac{dy}{dt}$

2. $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0.$$

10.2
2/7

<u>Ex1:</u> Sketch a graph of the curve parametrically given by
$$x(t) = 10 - t^2 \quad \text{and} \quad y = t^3 - 12t.$$

<u>x-intercepts:</u> $(y=0)$

solve $0 = t^3 - 12t$
$\qquad = t(t+\sqrt{12})(t-\sqrt{12})$.
$\qquad \Rightarrow t = 0$ or $t = \pm\sqrt{12}$

$t = 0 \Rightarrow (10,0)$
$t = \sqrt{12} \Rightarrow (-2,0)$
$t = -\sqrt{12} \Rightarrow (-2,0)$

<u>y-intercepts</u> $(x=0)$

solve $0 = 10 - t^2$
$\qquad \Rightarrow t = \pm\sqrt{10}$

$t = \sqrt{10} \Rightarrow (0, -2\sqrt{10})$
$t = -\sqrt{10} \Rightarrow (0, 2\sqrt{10})$

<u>slope</u> undefined: $(x'(t) = 0)$

solve $0 = -2t$
$\qquad \Rightarrow t = 0$

$t = 0 \Rightarrow (10,0)$

<u>slope = 0</u> $(y'(t) = 0)$

solve $0 = 3t^2 - 12$
$\qquad = 3(t^2 - 4)$
$\qquad \Rightarrow t = \pm 2$

$t = 2 \Rightarrow (6, -16)$
$t = -2 \Rightarrow (6, 16)$

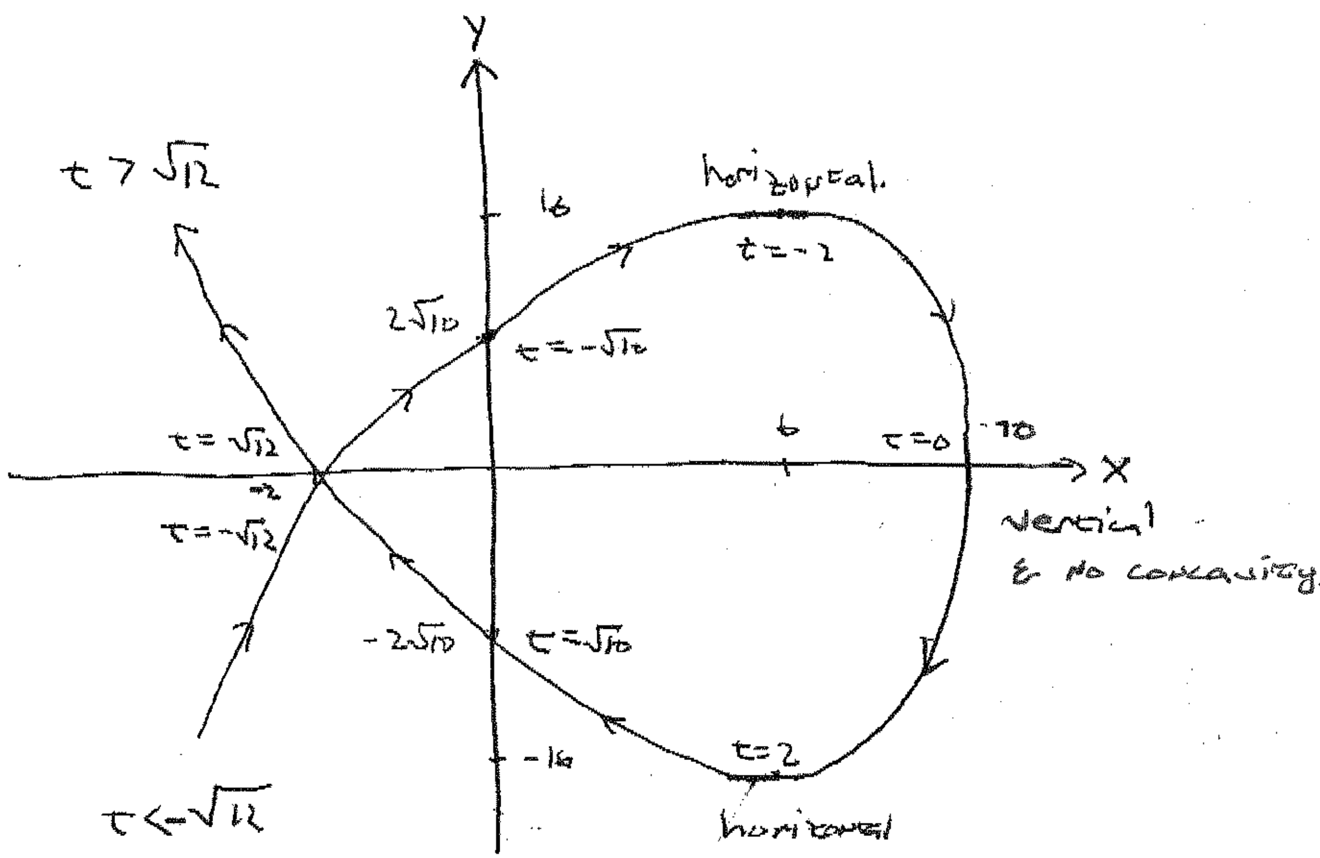
Find the slopes at $t = \pm\sqrt{12}$

$$\frac{dy}{dx} = \frac{3t^2 - 12}{-2t}$$

$$= -\frac{3}{2}\left(\frac{t^2-4}{t}\right)$$

$t = \sqrt{12} \Rightarrow m < 0$

$t = -\sqrt{12} \Rightarrow m > 0$



$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(-\frac{3}{2}t + \frac{6}{t}\right)}{\frac{dx}{dt}}$$

$$= \frac{-\frac{3}{2} - \frac{6}{t^2}}{-2t}$$

$$= \frac{3}{4t} + \frac{3}{t^3}$$

$$= \frac{3(t^3+4t)}{2t^3}$$

$$= \frac{3t(t^2+4)}{2t^3}$$

$$= \frac{3(t^2+4)}{2t^2}$$

The area "under" $(x(t), y(t))$ on $\alpha \leq t \leq \beta$ if the curve is traversed exactly once is given by

$$\boxed{3} \quad A = \int_{t=\alpha}^{t=\beta} y\,dx = \int_{t=\alpha}^{t=\beta} y(t)\,x'(t)\,dt.$$

($t=\alpha$ assumed to be on the left).

Ex 2: Find the area of a circle.

Let $x = \cos\theta$ and $y = \sin\theta$ on $0 \leq \theta \leq 2\pi$

$$A = \int_0^{2\pi} y(\theta)\,x'(\theta)\,d\theta$$

$$= \int_0^{2\pi} \sin(\theta)\,(-\sin(\theta))\,d\theta$$

$$= \int_{2\pi}^{0} \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_{2\pi}^{0} (1 - \cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{2\pi}^{0}.$$

$= -2\pi \longleftarrow$ negative since $\theta = 0$ is right of $\theta = 2\pi$

For the right sign, $A = 2\int_{\pi}^{0} y(\theta)\,x'(\theta)\,d\theta$

10.2
6/7

If a curve C w/ pts $(x(t), y(t))$ on $\alpha \le t \le \beta$ where x' and y' are cont. on $[\alpha, \beta]$ and C is traversed exactly once on $[\alpha, \beta]$, then the arclength of C is

$$\boxed{4} \quad L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex 3: Find the circumference of a circle w/ radius R.

$$x(\theta) = R\cos\theta \quad \text{and} \quad y(\theta) = R\sin\theta$$

$$x'(\theta) = -R\sin\theta \quad \text{and} \quad y'(\theta) = R\cos\theta$$

$$L = \int_0^{2\pi} \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} R\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} R\, d\theta$$

$$= \Big[R\theta\Big]_0^{2\pi}$$

$$= 2\pi R$$

If the curve given by $(x(t), y(t))$ on $\alpha \le t \le \beta$ is rotated about the x-axis, where x' and y' are cont. and $y(t) \ge 0$ on $[\alpha, \beta]$, then the resulting surface area is.

$\boxed{5}$ $$SA = \int_{\alpha}^{\beta} 2\pi\, y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex 4: Find the SA of the sphere w/ radius r.

$$SA = \int_0^{\pi} 2\pi r \sin\theta \, d\theta$$

$$= \Big[-2\pi r \cos\theta\Big]_0^{\pi}$$

$$= -2\pi r\,[-1 - (1)]$$

$$= 4\pi r \,//.$$