4.3 LAWS OF LOGARITHMS

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms. These properties give logarithmic functions a wide range of applications, as we will see in Section 4.5.

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let $A > 0$, $B > 0$, and C be any real numbers.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

■ **Proof** We make use of the property $\log_a a^x = x$ from Section 4.2.

Law 1. Let $\qquad \log_a A = u \qquad \text{and} \qquad \log_a B = v$

When written in exponential form, these equations become

$$a^u = A \qquad \text{and} \qquad a^v = B$$

Thus
$$\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})$$
$$= u + v = \log_a A + \log_a B$$

Law 2. Using Law 1, we have

$$\log_a A = \log_a\left[\left(\frac{A}{B}\right)B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$$

so
$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Law 3. Let $\log_a A = u$. Then $a^u = A$, so

$$\log_a(A^C) = \log_a(a^u)^C = \log_a(a^{uC}) = uC = C \log_a A$$

□

John Napier (1550–1617) was a Scottish landowner for whom mathematics was a hobby. We know him today because of his key invention—logarithms, which he published in 1614 under the title *A Description of the Marvelous Rule of Logarithms*. In Napier's time, logarithms were used exclusively for simplifying complicated calculations. For example, to multiply two large numbers we would write them as powers of 10. The exponents are simply the logarithms of the numbers. For instance,

$$\begin{aligned} 4532 \times 57783 &\approx 10^{3.65629} \times 10^{4.76180} \\ &= 10^{8.41809} \\ &\approx 261{,}872{,}564 \end{aligned}$$

The idea is that multiplying powers of 10 is easy (we simply add their exponents). Napier produced extensive tables giving the logarithms (or exponents) of numbers. Since the advent of calculators and computers, logarithms are no longer used for this purpose. The logarithmic functions, however, have found many applications, some of which are described in this chapter.

Napier wrote on many topics. One of his most colorful works is a book entitled *A Plaine Discovery of the Whole Revelation of Saint John*, in which he predicted that the world would end in the year 1700.

As the following examples illustrate, these laws are used in both directions. Since the domain of any logarithmic function is the interval $(0, \infty)$, we assume that all quantities whose logarithms occur are positive.

EXAMPLE 1 ■ Using the Laws of Logarithms to Expand Expressions

Use the Laws of Logarithms to rewrite each expression.

(a) $\log_2(6x)$ (b) $\log \sqrt{5}$

(c) $\log_5(x^3y^6)$ (d) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

SOLUTION

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ Law 1

(b) $\log \sqrt{5} = \log 5^{1/2} = \frac{1}{2}\log 5$ Law 3

(c) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3\log_5 x + 6\log_5 y$ Law 3

(d) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3

EXAMPLE 2 ■ Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$ (c) $-\frac{1}{3}\log 8$

SOLUTION

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $4^3 = 64$

(b) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ Law 2

$= \log_2 16 = 4$ Because $2^4 = 16$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ Law 3

$= \log\left(\frac{1}{2}\right)$ Property of negative exponents

≈ -0.301 Use a calculator

EXAMPLE 3 ■ Writing an Expression as a Single Logarithm

Express $3\log x + \frac{1}{2}\log(x + 1)$ as a single logarithm.

SOLUTION

$$\begin{aligned} 3 \log x + \tfrac{1}{2} \log(x+1) &= \log x^3 + \log(x+1)^{1/2} && \text{Law 3} \\ &= \log\left(x^3(x+1)^{1/2}\right) && \text{Law 1} \end{aligned}$$

■

EXAMPLE 4 ■ Writing an Expression as a Single Logarithm

Express $3 \ln s + \frac{1}{2} \ln t - 4 \ln(t^2 + 1)$ as a single logarithm.

SOLUTION

$$\begin{aligned} 3 \ln s + \tfrac{1}{2} \ln t - 4 \ln(t^2+1) &= \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4 && \text{Law 3} \\ &= \ln(s^3 t^{1/2}) - \ln(t^2+1)^4 && \text{Law 1} \\ &= \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right) && \text{Law 2} \end{aligned}$$

■

WARNING Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference.* For instance,

$$\log_a(x+y) \neq \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$.

Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right)$$

$$(\log_2 x)^3 \neq 3 \log_2 x$$

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base a, of each side.

$$\begin{aligned} b^y &= x && \text{Exponential form} \\ \log_a(b^y) &= \log_a x && \text{Take } \log_a \text{ of each side} \\ y \log_a b &= \log_a x && \text{Law 3} \\ y &= \frac{\log_a x}{\log_a b} && \text{Divide by } \log_a b \end{aligned}$$

This proves the following formula.

We may write the Change of Base Formula as

$$\log_b x = \left(\frac{1}{\log_a b}\right)\log_a x$$

So, $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\frac{1}{\log_a b}$.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if we put $x = a$, then $\log_a a = 1$ and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms and then using a calculator.

EXAMPLE 5 ■ Using the Change of Base Formula to Evaluate Logarithms

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct to five decimal places.

(a) $\log_8 5$ (b) $\log_9 20$

SOLUTION

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

■

EXAMPLE 6 ■ Using the Change of Base Formula to Graph a Logarithmic Function

Use a graphing calculator to graph $f(x) = \log_6 x$.

SOLUTION

Calculators don't have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

Since calculators do have an LN key, we can enter this new form of the function and graph it. The graph is shown in Figure 1. ■

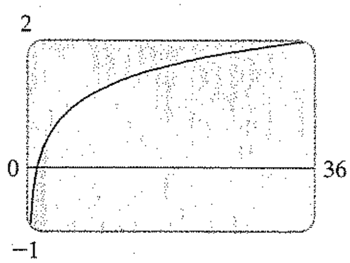

FIGURE 1

$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$

4.3 EXERCISES

1–26 ■ Use the Laws of Logarithms to rewrite the expression in a form with no logarithm of a product, quotient, root, or power.

1. $\log_2 (2x)$

2. $\log_3 (5y)$

3. $\log_2(x(x-1))$

4. $\log_5\left(\frac{x}{2}\right)$

5. $\log 6^{10}$

6. $\ln(\sqrt{z})$

7. $\log_2(AB^2)$

8. $\log_6 \sqrt[4]{17}$

9. $\log_3(x\sqrt{y})$

10. $\log_2(xy)^{10}$

11. $\log_5 \sqrt[3]{x^2+1}$

12. $\log_a\left(\frac{x^2}{yz^3}\right)$

13. $\ln \sqrt{ab}$

14. $\ln \sqrt[3]{3r^2 s}$

15. $\log\left(\frac{x^3 y^4}{z^6}\right)$

16. $\log\left(\frac{a^2}{b^4\sqrt{c}}\right)$

17. $\log_2\left(\frac{x(x^2+1)}{\sqrt{x^2-1}}\right)$

18. $\log_5 \sqrt{\frac{x-1}{x+1}}$

19. $\ln\left(x\sqrt{\frac{y}{z}}\right)$

20. $\ln \frac{3x^2}{(x+1)^{10}}$

21. $\log \sqrt[4]{x^2+y^2}$

22. $\log\left(\frac{x}{\sqrt[3]{1-x}}\right)$

23. $\log \sqrt{\frac{x^2+4}{(x^2+1)(x^3-7)^2}}$

24. $\log \sqrt{x\sqrt{y\sqrt{z}}}$

25. $\ln\left(\frac{x^3\sqrt{x-1}}{3x+4}\right)$

26. $\log\left(\frac{10^x}{x(x^2+1)(x^4+2)}\right)$

27–38 ■ Evaluate the expression.

27. $\log_5 \sqrt{125}$

28. $\log_2 112 - \log_2 7$

29. $\log 2 + \log 5$

30. $\log \sqrt{0.1}$

31. $\log_4 192 - \log_4 3$

32. $\log_{12} 9 + \log_{12} 16$

33. $\ln 6 - \ln 15 + \ln 20$

34. $e^{3\ln 5}$

35. $10^{2\log 4}$

36. $\log_2 8^{33}$

37. $\log(\log 1000^{10{,}000})$

38. $\ln(\ln(\ln e^{e^{200}}))$

39–48 ■ Rewrite the expression as a single logarithm.

39. $\log_3 5 + 5\log_3 2$

40. $\log 12 + \frac{1}{2}\log 7 - \log 2$

41. $\log_2 A + \log_2 B - 2\log_2 C$

42. $\log_5(x^2-1) - \log_5(x-1)$

43. $4\log x - \frac{1}{3}\log(x^2+1) + 2\log(x-1)$

44. $\ln(a+b) + \ln(a-b) - 2\ln c$

45. $\ln 5 + 2\ln x + 3\ln(x^2+5)$

46. $2(\log_5 x + 2\log_5 y - 3\log_5 z)$

47. $\frac{1}{3}\log(2x+1) + \frac{1}{2}[\log(x-4) - \log(x^4 - x^2 - 1)]$

48. $\log_a b + c\log_a d - r\log_a s$

49–56 ■ Use the Change of Base Formula and a calculator to evaluate the logarithm, correct to six decimal places. Use either natural or common logarithms.

49. $\log_2 5$

50. $\log_5 2$

51. $\log_3 16$

52. $\log_6 92$

53. $\log_7 2.61$

54. $\log_6 532$

55. $\log_4 125$

56. $\log_{12} 2.5$

57. Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

58. Draw graphs of the family of functions $y = \log_a x$ for $a = 2, e, 5$, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

59. Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

60. Simplify: $(\log_2 5)(\log_5 7)$

61. Show that $-\ln\left(x - \sqrt{x^2-1}\right) = \ln\left(x + \sqrt{x^2-1}\right)$.

DISCOVERY • DISCUSSION

62. Is the Equation an Identity? Discuss each equation and determine whether it is true for all possible values of the variables. (Ignore values of the variables for which any term is undefined.)

(a) $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

Section 4.3 ■ page 363

1. $1 + \log_2 x$ **3.** $\log_2 x + \log_2 (x - 1)$ **5.** $10 \log 6$
7. $\log_2 A + 2 \log_2 B$ **9.** $\log_3 x + \frac{1}{2} \log_3 y$
11. $\frac{1}{3} \log_5(x^2 + 1)$ **13.** $\frac{1}{2}(\ln a + \ln b)$
15. $3 \log x + 4 \log y - 6 \log z$
17. $\log_2 x + \log_2(x^2 + 1) - \frac{1}{2} \log_2(x^2 - 1)$
19. $\ln x + \frac{1}{2}(\ln y - \ln z)$ **21.** $\frac{1}{4} \log(x^2 + y^2)$
23. $\frac{1}{2}[\log(x^2 + 4) - \log(x^2 + 1) - 2 \log(x^3 - 7)]$
25. $3 \ln x + \frac{1}{2} \ln (x - 1) - \ln (3x + 4)$
27. $\frac{3}{2}$ **29.** 1 **31.** 3 **33.** $\ln 8$ **35.** 16 **37.** $4 + \log 3$
39. $\log_3 160$ **41.** $\log_2(AB/C^2)$ **43.** $\log\left[\dfrac{x^4(x-1)^2}{\sqrt[3]{x^2+1}}\right]$
45. $\ln[5x^2(x^2 + 5)^3]$
47. $\log\left[\sqrt[3]{2x+1}\sqrt{(x-4)/(x^4-x^2-1)}\right]$
49. 2.321928 **51.** 2.523719 **53.** 0.493008
55. 3.482892
57.