16.9: The Divergence Thm.

NOTES: Positive orientation refers to the normals.
label of "flux"
withdrawal deadline: 12/4

Since it was <u>soooo</u> successful, let's try the same format as the last section.

recall: Green's Thm

$$\oint_C \overset{P\,dx + Q\,dy}{\vec{F} \cdot d\vec{r}} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

and one of its vector forms

$$\underbrace{\oint_C \vec{F} \cdot \vec{n}\, ds}_{\substack{\text{integral of} \\ \vec{F} \cdot \vec{n} \text{ around} \\ \text{a boundary}}} = \underbrace{\iint_D \operatorname{div} \vec{F}(x,y)\, dA}_{\substack{\text{integral of} \\ \operatorname{div} \vec{F} \text{ over} \\ \text{a region}}}$$

Div: Interp

if $\vec{F}$ gives the velocity of a fluid w/ constant density... $\operatorname{div}\vec{F}$ gives the mass of the fluid leaving the region per unit time per unit volume.

$$\Downarrow \qquad \qquad \Downarrow$$

$$\iint_S \vec{F} \cdot \vec{N}\, ds = \iiint_E \operatorname{div} \vec{F}(x,y,z)\, dV$$

<u>Divergence Thm:</u> Let E be a simple solid region & let S be the boundary surface of E, given w/ positive orientation. Let $\vec{F}$ be a vector field whose component fcts have cont. partials on an open region that contains E.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$\boxed{\begin{matrix}16.9\\ 2/5\end{matrix}}$

ex1: Calculate the flux of $\vec{F} = \langle x^2 z^3, 2xyz^3, xz^4 \rangle$ across the surface of the box w/vertices $(\pm 1, \pm 2, \pm 3)$

$$\operatorname{div} \vec{F} = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$$

$$\text{Flux} = \int_{-3}^{3} \int_{-2}^{2} \int_{-1}^{1} 8xz^3 \, dx \, dy \, dz = 0$$

$$\left[4x^2 z^3\right]_{-1}^{1}$$

ex2: Calculate the flux of $\vec{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$ across the surface S consisting of $z = \sqrt{1-x^2-y^2}$ & $x^2+y^2 \le 1$ on the xy plane.

$$\operatorname{div} \vec{F} = \left(\sqrt{x^2+y^2+z^2}\right)^{-1} + x \cdot \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot 2x + \ldots$$

$$= \left(\sqrt{x^2+y^2+z^2}\right)^{-1} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} + \ldots$$

$$= 3\left(\sqrt{x^2+y^2+z^2}\right)^{-1} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{3(x^2+y^2+z^2) - (x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

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3/5

$$\Rightarrow \text{Flux} = \iiint_E \frac{2}{\sqrt{x^2+y^2+z^2}}\, dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \frac{2}{\rho}\,\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \left[\rho^2\right]_0^1 \sin\phi \, d\theta\, d\phi$$

$$= \int_0^{\pi/2} 2\pi \sin\phi \, d\phi$$

$$= \left[-2\pi \cos\phi\right]_0^{\pi/2}$$

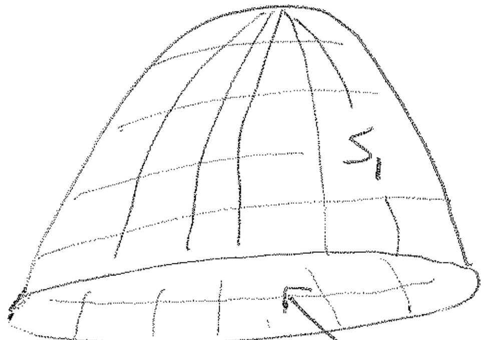
$$= -2\pi(0-1)$$

$$= 2\pi$$

$\boxed{\begin{matrix} 16.9 \\ 4/5 \end{matrix}}$

ex 3: recall from Test 3.

Find the flux of $\vec{F} = \langle -x, y, x^2+y^2 \rangle$ across $z = 9 - x^2 - y^2$ when $z \geq 0$.

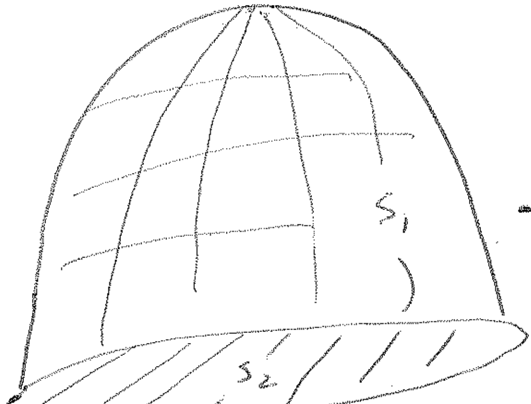


$\boxed{\text{Good news.}}$

$$\operatorname{div}(\vec{F}) = -1 + 1 = 0$$

$\boxed{\text{Bad news}}$

open @ bottom .... can't use the divergence Thm... directly.



$$\iint_{S_1 \cup S_2} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1 \cup S_2} \vec{F} \cdot d\vec{S} - \iint_{S_2} \vec{F} \cdot d\vec{S}$$

$$= \iiint_E \underbrace{\operatorname{div} \vec{F}}_{0} \, dV - \iint_{S_2} \vec{F} \cdot d\vec{S}$$

$$= -\iint_{S_2} \vec{F} \cdot d\vec{S}$$

parametrize S_2

$\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 0 \rangle$

$\vec{r}_R = \langle \cos\theta, \sin\theta, 0 \rangle$

$\vec{r}_\theta = \langle -R\sin\theta, R\cos\theta, 0 \rangle$

$\vec{r}_R \times \vec{r}_\theta = \langle 0, 0, +R \rangle$

orientation

$$= -\iint_{D_2} \langle -R\cos\theta, R\sin\theta, R^2 \rangle \cdot \langle 0, 0, -R \rangle \, dA$$

$$= +\int_0^3 \int_0^{2\pi} R^3 \, d\theta \, dR$$

$$= \left[\frac{2\pi R^4}{4} \right]_0^3 \quad \frac{81\pi}{2}$$

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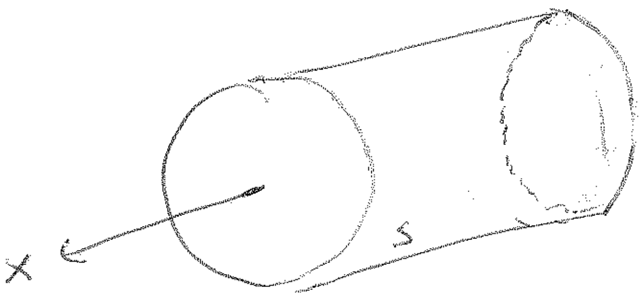
ex 4: Find the flux of $\vec{F} = \langle 3xy^2, xe^z, z^3 \rangle$ across the solid bounded by $y^2+z^2=1$, $x=-1$, & $x=2$.

$$\operatorname{div}(F) = 3y^2 + 3z^2$$



$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iiint 3(y^2+z^2)\,dV$$

$X = x$, $y = R\cos\theta$, $z = R\sin\theta$

$$= \int_0^1 \int_0^{2\pi} \int_{-1}^{2} 3R^2 \cdot R \, dx \, d\theta \, dR$$

$$= 18\pi \int_0^1 R^3 \, dR$$

$$= \frac{9}{2}\pi$$

21.6 MAXWELL'S EQUATIONS. A FINAL THOUGHT

To gain a slight glimpse of the significance of the ideas of this chapter, we look very briefly at the famous equations formulated in the 1860s by James Clerk Maxwell (1831–1879). These equations are remarkable because they contain a complete theory of everything that was then known or would later become known about electricity and magnetism. Maxwell was the greatest theoretical physicist of the nineteenth century, and an excellent account of his life and work is given by James R. Newman in *Science and Sensibility,* vol. 1, pp. 139–193 (Simon and Schuster, 1961).

In Maxwell's theory there are two vector fields defined at every point in space: an electric field **E** and a magnetic field **B**. The electric field is produced by charged particles (electrons, protons, etc.) that may be moving or stationary, and the magnetic field by moving charged particles.

All known phenomena involving electromagnetism can be explained and understood by means of *Maxwell's equations*:

1 $\nabla \cdot \mathbf{E} = \dfrac{q}{\epsilon_0}$.

2 $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$.

3 $\nabla \cdot \mathbf{B} = 0$.

4 $c^2 \nabla \times \mathbf{B} = \dfrac{\mathbf{j}}{\epsilon_0} + \dfrac{\partial \mathbf{E}}{\partial t}$.

Here q is the charge density, ϵ_0 is a constant, c is the velocity of light, and $\mathbf{j}$ is the current density (not to be confused with the unit vector in the direction of the y-axis). We make no attempt to discuss the meaning of these four equations, but we do point out that the first two make statements about the divergence and curl of **E**, and the second two about the divergence and curl of **B**. Equivalent verbal statements of Maxwell's equations are given by Richard Feynman (Nobel

Prize, 1965) on p. 18-2 in vol. 2 of his *Lectures on Physics* (Addison-Wesley, 1964):

1′ Flux of **E** through a closed surface $= \dfrac{\text{charge inside}}{\epsilon_0}$.

2′ Line integral of **E** around a loop $= -\dfrac{\partial}{\partial t}$ (flux of **B** through the loop).

3′ Flux of **B** through a closed surface $= 0$.

4′ c^2 (integral of **B** around a loop) $= \dfrac{\text{current through the loop}}{\epsilon_0} + \dfrac{\partial}{\partial t}$ (flux of **E** through the loop).

By a "loop," Feynman means what we have called a simple closed curve. The fact that these verbal statements are indeed equivalent to Maxwell's equations 1 to 4 depends on Gauss's Theorem and Stokes' Theorem. This is perhaps easier to grasp when these verbal statements are expressed in terms of line and surface integrals:

1″ $\displaystyle\iint_S \mathbf{E}\cdot\mathbf{n}\,dA = \frac{Q}{\epsilon_0}$ (S is a closed surface).

2″ $\displaystyle\oint_C \mathbf{E}\cdot d\mathbf{R} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\mathbf{n}\,dA$ (C is a simple closed curve and S is a surface bounded by C).

3″ $\displaystyle\iint_S \mathbf{B}\cdot\mathbf{n}\,dA = 0$ (S is a closed surface).

4″ $\displaystyle c^2\oint_C \mathbf{B}\cdot d\mathbf{R} = \frac{1}{\epsilon_0}\iint_S \mathbf{j}\cdot\mathbf{n}\,dA + \frac{\partial}{\partial t}\iint_S \mathbf{E}\cdot\mathbf{n}\,dA$ (C is a simple closed curve and S is a surface bounded by C).

Our only purpose in mentioning these matters is to try to make it perfectly clear to the student that the mathematics we have been doing in this chapter has profoundly important applications in physical science. Feynman devotes the first 21 chapters in vol. 2 of his *Lectures* to the meaning and implications of Maxwell's equations. At one point he memorably remarks:

> From a long view of the history of mankind—seen from, say, ten thousand years from now—there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.

In making this provocative comment, perhaps Feynman was carried away by his ebullient enthusiasm—but perhaps not.

Div. (Gauss') Thm

Gauss's Theorem is a profound theorem of mathematical analysis, with a wealth of important applications to many of the physical sciences. The cursory sketch of these ideas that we have given here—together with a similar sketch of Stokes' Theorem in the next section—is perhaps as far as an introductory calculus course should go in this direction. Students who wish to learn more are encouraged to continue and take advanced courses (vector analysis, potential theory, mathematical physics, etc.) in which these themes are fully developed.

Stokes's Thm.

One final remark: The relations among properties (a) through (d) will not be truly understood until we reach the stage at which the implications described above can be grasped as an organic whole and recalled in a few seconds of thought.

We have seen that Gauss's Theorem relates an integral over a closed surface to a corresponding volume integral over the region of space enclosed by the surface, and Stokes' Theorem relates an integral around a closed curve to a corresponding surface integral over any surface bounded by the curve. As we suggested at the beginning of Section 21.4, these statements are very similar and are presumably somehow connected with each other. It turns out that both are special cases of a powerful theorem of modern analysis called the *generalized Stokes Theorem*. Students who wish to understand these relationships must study the theory of differential forms.

Specifically, if **F** is a vector field defined in a simply connected region of space, then any one of the following four properties implies the remaining three:*

(a) $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ for every simple closed curve C.

(b) $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of the path.

(c) **F** is a gradient field, i.e., $\mathbf{F} = \nabla f$ for some scalar field f.

(d) curl $\mathbf{F} = \mathbf{0}$.

Simmons "Calc" ch 21, sec 4,5,6