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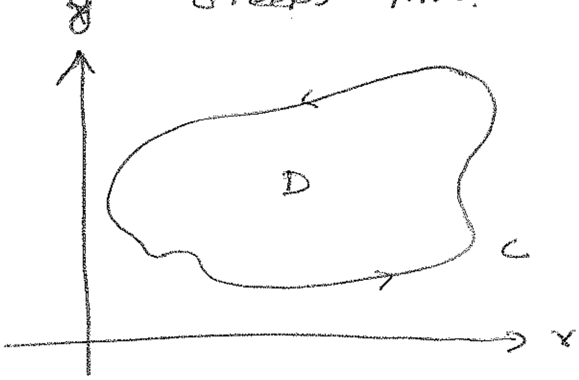
Stoke's Thm.

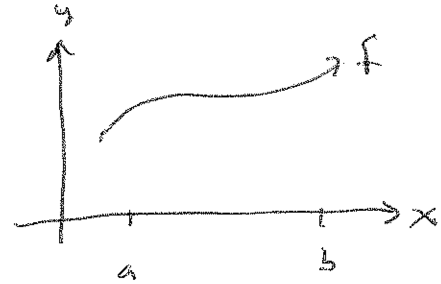
Green's Thm.



derivative $f = F'$

antiderivative on the boundary

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

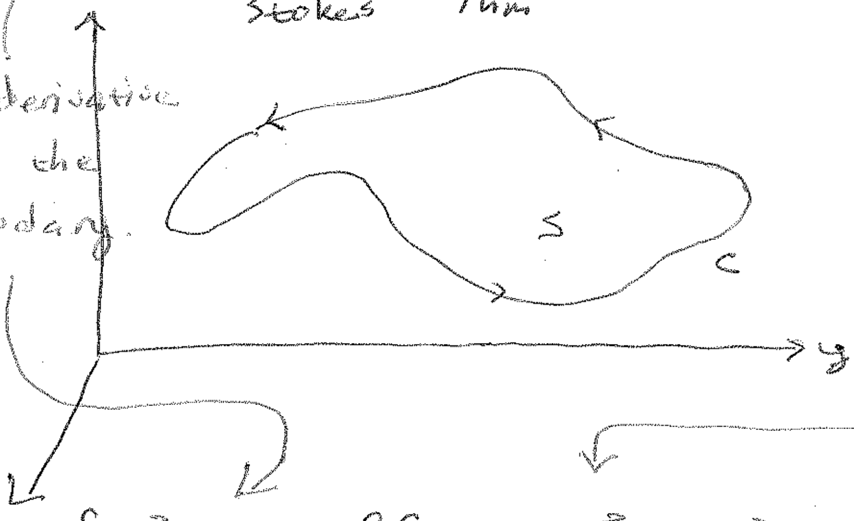


$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

Stoke's Thm

antiderivative on the boundary.

"Derivative" inside a region

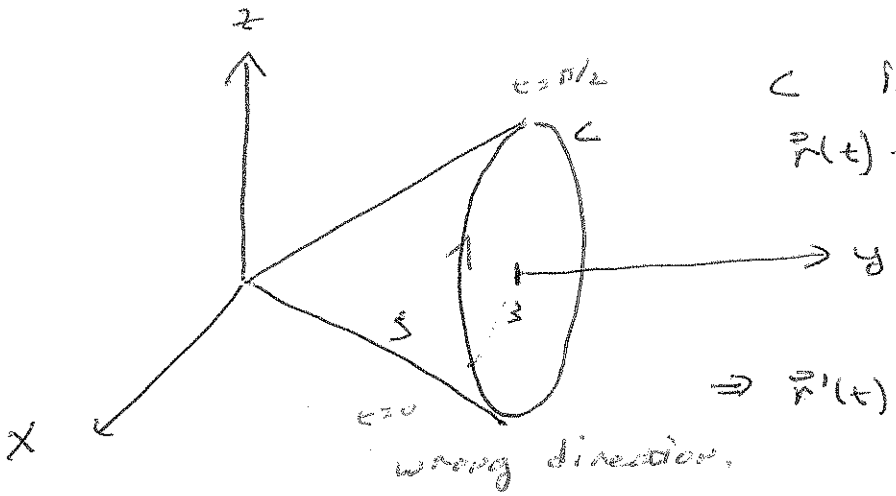


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$$

Stoke's Thm: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C w/positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$

<u>Ex 1:</u> If $\vec{F} = \langle x^2y^3z, \sin(xyz), xyz \rangle$

& S is the part of a cone $y^2 = x^2 + z^2$ on $0 \le y \le 3$ oriented in the direction of the positive y-axis, evaluate $I = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$



C is parametrized by

$\vec{r}(t) = \langle 3\cos t, 3, 3\sin t \rangle$

on $0 \le t \le 2\pi$

$\Rightarrow \vec{r}\,'(t) = \langle -3\sin t, 0, 3\cos t \rangle$

$$I = \oint_C \vec{F} \cdot d\vec{r} = -\int_0^{2\pi} F(\vec{r}(t))\, \vec{r}\,'(t)\, dt$$

$$= -\int_0^{2\pi} \langle 9\cos^2(t) \cdot 27 \cdot 3\sin t, \sin(27\sin t\cos t), 27\sin t\cos t \rangle \cdot \langle -3\sin t, 0, 3\cos t \rangle\, dt$$

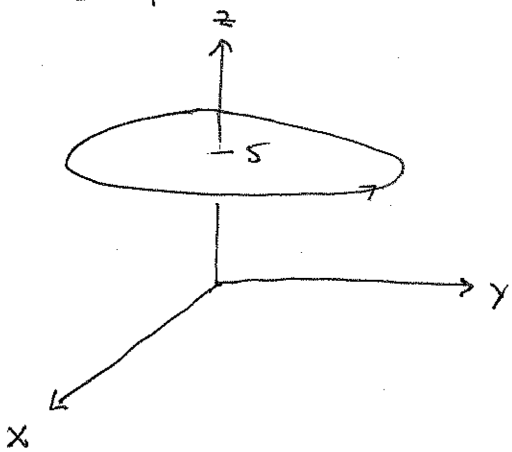
$$= -81 \int_0^{2\pi} -27\cos^2 t \sin^2 t + 0 + \sin t\cos^2 t \, dt$$

$$= -81 \left[\frac{-\cos^3 t}{3} \right]_0^{2\pi} + 2187 \left[\frac{\sin t \cos^3 t}{4} + \frac{1}{4} \cdot \int \cos^2 t\, dt \right]_0^{2\pi}$$

$$= +27(1 - 1) + \frac{2187}{4} \left(\sin t\cos^3 t + \frac{1}{2}t + \frac{1}{4}\sin 2t \right]_0^{2\pi}$$

$$= + \frac{2187}{4}\pi$$

Ex 2: If $\vec{F} = \langle yz, 2xz, e^{xy} \rangle$ and C is the circle $x^2 + y^2 = 16$ w/ $z = 5$ traversed ccw, find the work $\oint_C \vec{F} \cdot d\vec{r}$



(1) directly...

$\vec{r}(t) = \langle 4\cos t, 4\sin t, 5 \rangle$

$\vec{r}'(t) = \langle -4\sin t, 4\cos t, 0 \rangle \, dt$

$$W = \int_0^{2\pi} \Big[20\sin t \cdot -4\sin t + 40\cos t \cdot 4\cos t + e^{16\sin t\cos t} \cdot 0 \Big] \, dt$$

$$= 80 \int_0^{2\pi} (2\cos^2 t - \sin^2 t) \, dt$$

$$= 80 \int_0^{2\pi} (1 + \cos 2t) - \left(\frac{1 - \cos 2t}{2} \right) dt$$

$$= 40 \int_0^{2\pi} (2 + 2\cos 2t - 1 + \cos 2t) \, dt$$

$$= 40 \left[t + \frac{3}{2} \sin 2t \right]_0^{2\pi}$$

$$= 80\pi$$

(2) using stoke's thm.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix}$$

$$= \langle xe^{xy} - 2x, -ye^{xy}, 2z - z \rangle$$

parametrize S...

$\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 5 \rangle$

on $0 \le R \le 4$ and $0 \le \theta \le 2\pi$

$\vec{r}_R = \langle \cos\theta, \sin\theta, 0 \rangle$

$\vec{r}_\theta = \langle -R\sin\theta, R\cos\theta, 0 \rangle$

$\vec{r}_R \times \vec{r}_\theta = \langle 0, 0, R \rangle$

$$\text{work} = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \iint_S \text{curl } \vec{F} \cdot (\vec{r}_R \times \vec{r}_\theta) \, dA$$

$$= \int_0^4 \int_0^{2\pi} 5R \, d\theta \, dR$$

$$= \int_0^4 10\pi R \, dR$$

$$= \left[5\pi R^2 \right]_0^4$$

$$= 80\pi$$

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<u>key point</u> w/ the given conditions, Stoke's thm says that $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ does not depend upon S. $\therefore$ That is

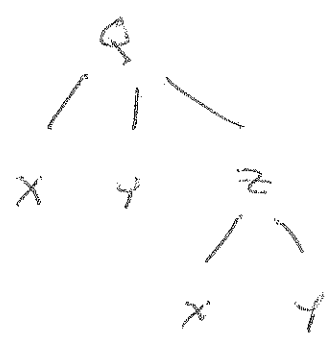
$$\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$$

if C is the boundary of both S_1 & S_2.

- If $\vec{F}$ is a velocity field in $\mathbb{R}^3$, then $\oint_C \vec{F}\cdot d\vec{r}$ gives the circulation of $\vec{F}$ about C.

Recall:
- Stoke's Thm: If S is a surface in space w/ boundary curve C, then the circulation of a vector field $\vec{F}$ around C is equal to the integral over S of the normal component of the curl of $\vec{F}$: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$

(A) multivariate chain rule



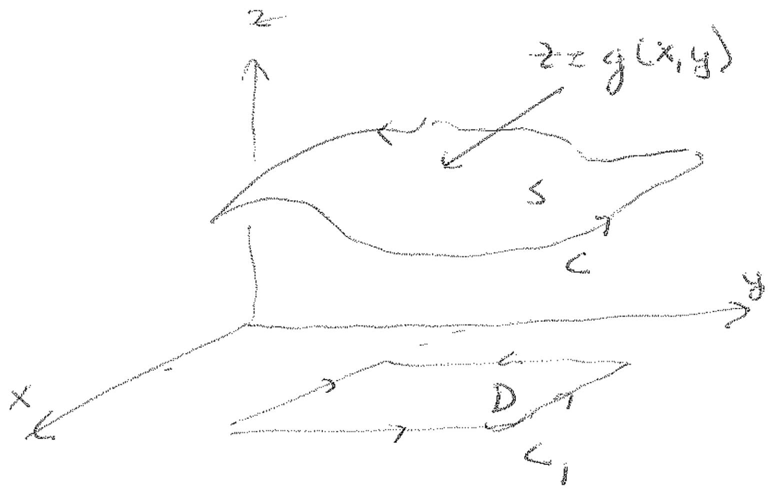
$$\frac{\partial}{\partial x} Q = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\cdot\frac{\partial z}{\partial y}$$

(B) Green's Thm.

(i) $\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$

(ii) $\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\vec{k}\,dA$

If S is a graph & $\vec{F}$, S, & C are nice.



S is $z = g(x,y)$ where $(x,y) \in D$.

$C = g(C_1)$ (notation!)

C & C_1 have pos. orientation

$\vec{F} = \langle P, Q, R \rangle$ where the partials are cont.

<u>claim</u>: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$

☐ Go thru proof in the book. (Tons of references)

<u>claim</u>: If $\text{curl}\,\vec{F} = \vec{0}$ on $\mathbb{R}^3$, then $\vec{F}$ is conservative.

☐ proof.

Assume $\text{curl}\,\vec{F} = \vec{0}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} \quad \text{(Stoke's Thm)}$$

$$= \iint_S \vec{0} \cdot d\vec{S} \quad \text{(by assumption)}$$

$$= 0$$

So, $\text{curl}\,\vec{F} = \vec{0} \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \Rightarrow \vec{F}$ is conservative

So a conservative vector field is irrotational.

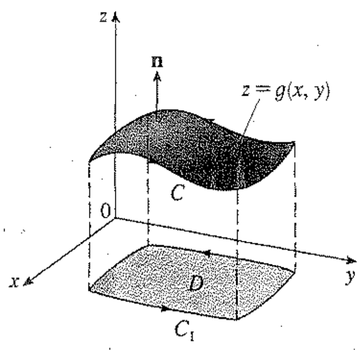


FIGURE 2

PROOF OF A SPECIAL CASE OF STOKES' THEOREM We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. (See Figure 2.) We are also given that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply Formula 16.7.10 with $\mathbf{F}$ replaced by curl $\mathbf{F}$. The result is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$\boxed{2} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of C_1, then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g\big(x(t), y(t)\big) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

F $z = g(x, y)$

line to iterated integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt$$

$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt$$

$$= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt$$

describe as a line integral

$$= \int_{C_1} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy$$

Green's Thm

$$= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z and that z is itself a function of x and y, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\cancel{\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}} + \cancel{R\frac{\partial^2 z}{\partial x\,\partial y}}\right)\right.$$

$$\left. - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\cancel{\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}} + \cancel{R\frac{\partial^2 z}{\partial y\,\partial x}}\right)\right] dA$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \qquad \blacksquare$$