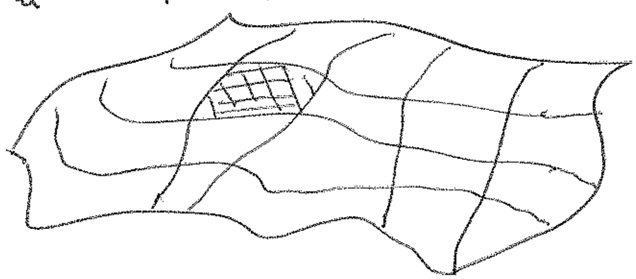
16.7: Surface Integrals.

Recall that we can parametrize a surface S in $\mathbb{R}^3$ as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

where $(u,v) \in D$.

Last section, we saw that the SA of our surface is $A = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \iint_S dS$

where $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u_i \Delta v_j$ is the area of the patch.



At each pt (x,y,z) on S, we could assign a value $f(x,y,z)$. This could be the density of a contoured sheet or perhaps the ~~total~~ rainfall on a hillside.

$$\iint_S f(x,y,z)\, dS = \lim_{\Delta S \to 0} \sum \sum f(x,y,z) \Delta S,$$

we know from 16.6, that $\Delta S = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

so
$$\underbrace{\iint_S f(x,y,z)\, dS}_{\text{WRT to SA}} = \underbrace{\iint_D f(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| \, dA.}_{\text{WRT the parameters}}$$

Ex1: $I = \iint_S yz\,dS$, S is the surface w/ parametric equations $\vec{r}(u,v) = \langle u^2, u\sin v, u\cos v\rangle$ on $0 \le u \le 1$, $0 \le v \le \pi/2$.

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$$\vec{r}(u,v) = \langle u^2, u\sin v, u\cos v\rangle \qquad 0 \le u \le 1$$

$$0 \le v \le \frac{\pi}{2}$$

$$\vec{r}_u = \langle 2u, \sin v, \cos v\rangle$$

$$\vec{r}_v = \langle 0, u\cos v, -u\sin v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix}$$

$$= \langle \underbrace{-u\sin^2 v - u\cos^2 v}_{-u.}, 2u^2\sin v, 2u^2\cos v\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 + 4u^4\sin^2 v + 4u^4\cos^2 v}$$

$$= u\sqrt{1+4u^2}$$

$$f(x,y,z) = yz$$

AND $f(\vec{r}(u,v)) = u\sin v \cdot u\cos v$

$$= u^2 \sin v \cos v$$

SO $$I = \int_0^{\frac{\pi}{2}}\int_0^1 u^3 \sin v\cos v\sqrt{1+4u^2}\,du\,dv$$

$$= \int_0^{\frac{\pi}{2}} \sin v\cos v\,dv \cdot \int_0^1 u^3\sqrt{1+4u^2}\,du.$$

$$= \left[\frac{\sin^2 v}{2}\right]_0^{\pi/2} \cdot \int_1^5 \frac{1}{8}\cdot\left(\frac{w-1}{4}\right)\sqrt{w}\,dw$$

$$= \frac{1}{2} \cdot \frac{1}{32}\int_1^5 w^{3/2} - w^{1/2}\,dw$$

$$= \frac{1}{2} \cdot \frac{1}{32}\left[\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right]_1^5$$

$$= \frac{1}{64}\left[\frac{2}{5}\cdot 5^{5/2} - \frac{2}{3}\cdot 5^{3/2} - \frac{2}{5} + \frac{2}{3}\right]$$

$$= \frac{5\sqrt{5}}{48} + \frac{1}{240}$$

Derivation when S is the graph of a fct $x = g(y,z)$. Parametrize ... (used in later examples).

$$\vec{r}(y,z) = \langle g(y,z), y, z\rangle$$

$$\vec{r}_y = \langle g_y(y,z), 1, 0\rangle$$

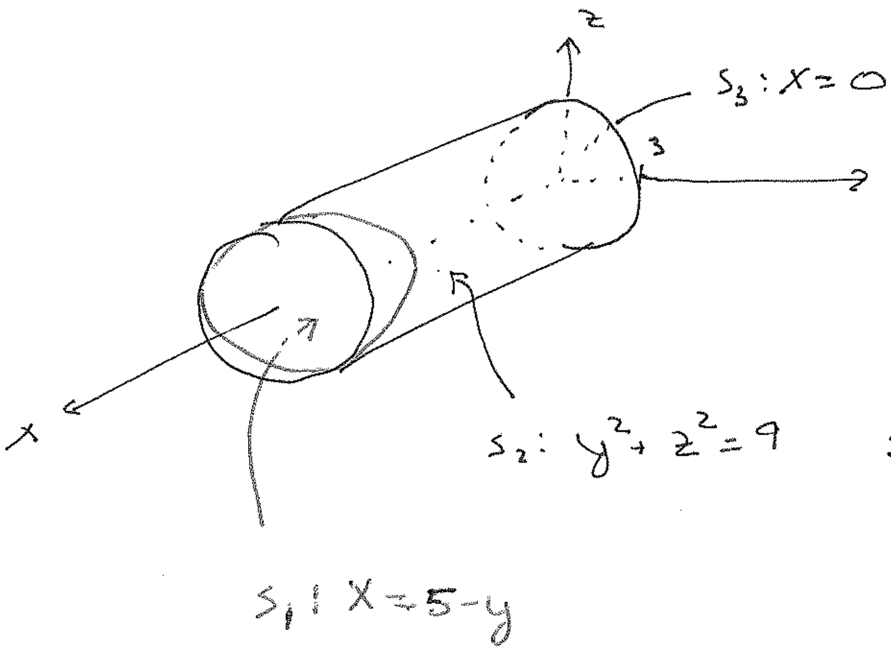
$$\vec{r}_z = \langle g_z(y,z), 0, 1\rangle$$

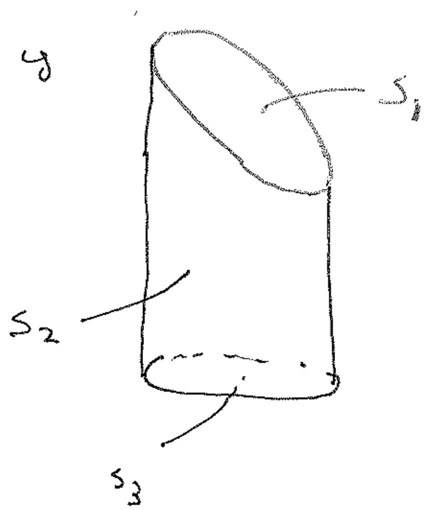
$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial g}{\partial y} & 1 & 0 \\ \frac{\partial g}{\partial z} & 0 & 1 \end{vmatrix} = \left\langle 1, -\frac{\partial g}{\partial y}, -\frac{\partial g}{\partial z}\right\rangle$$

w/norm: $\sqrt{1 + (g_y)^2 + (g_z)^2}$

AND $\iint_S f(x,y,z)\,dS = \iint_D f(g(y,z), y, z)\sqrt{1 + (g_y)^2 + (g_z)^2}\,dA$

Ex 2





If the figure represents S, find $\iint_S xy\,dS$

$$I = \iint_S xz\,dS = \iint_{S_1} xz\,dS + \iint_{S_2} xz\,dS + \underbrace{\iint_{S_3} xz\,dS}_{I_3 = 0 \text{ since } x=0 \text{ on } S_3.}$$

S_2: $\vec{r}(u, \theta) = \langle u, 3\cos\theta, 3\sin\theta \rangle$

$\vec{r}_u = \langle 1, 0, 0 \rangle$

$\vec{r}_\theta = \langle 0, -3\sin\theta, 3\cos\theta \rangle$

$$\vec{r}_u \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -3\sin\theta & 3\cos\theta \end{vmatrix} = \langle 0, -3\cos\theta, -3\sin\theta \rangle$$

$|\vec{r}_u \times \vec{r}_\theta| = 3$

so $I_2 = \iint_{S_2} xz\,dS = \int_0^{2\pi} \int_0^{5-3\cos\theta} u \cdot 3\sin\theta \cdot 3\,du\,d\theta$

$\boxed{\begin{matrix}16.7\\5/9\end{matrix}}$

$\hookrightarrow I_2 = 9 \int_0^{2\pi} \left[\frac{u^2}{2} \right]_0^{5-3\cos\theta} \sin\theta \, d\theta$

$= 9 \int_0^{2\pi} \frac{(5-3\cos\theta)^2}{2} \sin\theta \, d\theta$

Let $w = \cos\theta \Rightarrow -dw = \sin\theta \, d\theta$

$= -9 \int_1^1 \frac{(5-3w)^2}{2} dw$

$= 0$

S_3 : $X = 5 - y$ OR $\vec{r}(y,z) = \langle 5-y, y, z \rangle$

$I_3 = \iint_{S_3} xz \, ds = \iint_D (5-y) \cdot z \sqrt{1 + 1^2 + 0^2} \, dA$

$D \quad y^2 + z^2 \leq 9$

$= \sqrt{2} \iint_D 5z - zy \, dA$

$y = r\cos\theta \quad z = r\sin\theta$

$= \sqrt{2} \int_0^{2\pi} \int_0^3 (5r\sin\theta - r^2\sin\theta\cos\theta) \cdot r \, dr \, d\theta$

$= \sqrt{2} \int_0^{2\pi} \left[\frac{5r^3}{3} \sin\theta - \frac{r^4}{4} \sin\theta\cos\theta \right]_0^3 d\theta$

$= \sqrt{2} \int_0^{2\pi} \left(45\sin\theta - \frac{81}{4} \sin\theta\cos\theta \right) d\theta$

$= 0.$

Oriented surfaces

- mobius strip
- which way is "up"?
  - Graph
  - closed surface
- $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ (surface given parametrically by $\vec{r}(u,v)$)
- $\vec{n} = \dfrac{\langle 1, -g_y, -g_z \rangle}{\sqrt{1+(g_y)^2+(g_z)^2}}$ (if $x = g(y,z)$)

Surface Integrals of Vector Fields

Defn: If $\vec{F}$ is a cont. vector field defined on an oriented surface S w/ unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

this is the flux of $\vec{F}$ across S.

Fish net analogy: The flux gives the rate of flow of a liquid w/ density 1 in units of mass/time.

to simplify calculations ... If S is given by $\vec{r}(u,v)$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, dS$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA$$

"volume" of a parallel piped.

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

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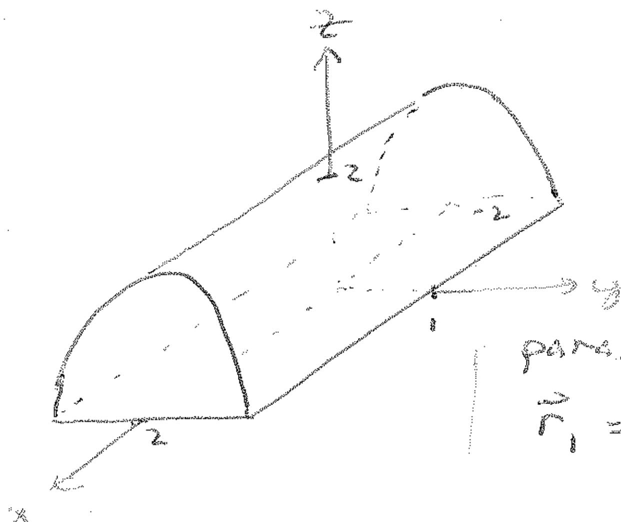
Ex 5: Set up an integral to find the flux of $\vec{F} = \langle \sin(xyz), x^2 y, z^2 e^{x/5} \rangle$ across $4y^2 + z^2 = 4$ where $z \geq 0$ & $-2 \leq x \leq 2$.

$z = \pm\sqrt{4-4y^2}$



parametrize.

$\vec{r}_1 = \langle u, \cos\theta, 2\sin\theta \rangle$

on $-2 \leq u \leq 2$ & $0 \leq \theta \leq \pi$

or $\vec{r}_2 = \langle x, y, 2\sqrt{1-y^2} \rangle$

we will use $\vec{r}_2$ since S is the graph of a fct.

$\vec{r}_x = \langle 1, 0, 0 \rangle$

$\vec{r}_y = \langle 0, 1, \frac{-2y}{\sqrt{1-y^2}} \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 0, \frac{2y}{\sqrt{1-y^2}}, 1 \rangle$

$$\text{Flux} = \int_{-1}^{1}\int_{2}^{2} \langle \sin(xy \cdot 2\sqrt{1-y^2}), x^2 y, 4(1-y^2)e^{x/5} \rangle \cdot \langle 0, \frac{2y}{\sqrt{1-y^2}}, 1 \rangle \, dx\, dy$$

$$= \int_{-1}^{1}\int_{2}^{2} \frac{2x^2y^2}{\sqrt{1-y^2}} + 4(1-y^2)e^{x/5} \, dx\, dy.$$

$$= \frac{1}{3}(16\pi + 80e^{2/5} - 80e^{-2/5})$$

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Find the flux across the graph of a fct.

If $\vec{F} = \langle P, Q, R \rangle$ & $S: x = g(y,z)$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$
$$= \iint_D \vec{F} \cdot (\vec{r}_y \times \vec{r}_z)\, dA$$

where $\vec{r}(y,z) = \langle g(y,z), y, z \rangle$ on $(y,z) \in D$ is a parametrization for S.

$$= \iint_D \langle P, Q, R \rangle \cdot \langle 1, -g_y, -g_z \rangle\, dA$$

$$= \iint_D \left(P - Q \frac{\partial g}{\partial y} - R \frac{\partial g}{\partial z} \right) dA.$$

Ex4: Set up an integral to evaluate

$I = \iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, -z, y \rangle$

and S is $x^2 + y^2 + z^2 = 4$ in the 1st octant w/ orientation toward the origin.

soln: $x = \sqrt{4 - y^2 - z^2}$ on $0 \le y \le 2$ and $0 \le z \le \sqrt{4 - y^2}$

orientation

$$\text{Flux} = -\iint x - (-z) \frac{-y}{\sqrt{4-y^2-z^2}} - y \frac{-z}{\sqrt{4-y^2-z^2}}\, dA.$$

$$= -\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4 - y^2 - z^2} - \frac{yz}{\sqrt{4-y^2-z^2}} + \frac{yz}{\sqrt{4-y^2-z^2}}\, dz\, dy$$

$$= -\frac{4}{3}\pi$$