16.3: Fundamental Thm for Line Integrals.

The Fundamental Theorem for Line Integrals

Thm: Let c be a smooth curve given by the vector fct $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable fct of 2 or 3 variables whose gradient ∇f is cont. on C. Then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Pt: If your integrand is a gradient field, then you can evaluate the integral w/ the end pts ... irrespective of path ... no parametrization needed.

$\square$ proof.
$$\int_C \vec{\nabla f} \cdot d\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \Big[\int \underbrace{\vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\#2}\, dt \Big]_a^b$$

$$= \Big[\int \underbrace{\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right)}_{\#3} dt \Big]_a^b$$

$$= \Big[\int \underbrace{\frac{d}{dt} f(\vec{r}(t))}_{\#1}\, dt \Big]_a^b$$

$$= \Big[f(\vec{r}(t)) \Big]_a^b$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

by analogy...;

$$\underbrace{\frac{d}{dt} f(x(t))}_{\#1} = \underbrace{f'(x(t))\, x'(t)}_{\#2} = \underbrace{\frac{df}{dx} \cdot \frac{dx}{dt}}_{\#3}$$

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ex1: Find the work done by a force field $\vec{F} = \langle 2y^{3/2}, 3x\sqrt{y} \rangle$ along the curve $y = x^2$ on $1 \le x \le 2$.

Solution: Let's assume that $\exists f$ st. $\vec{\nabla} f = \vec{F}$. If we can verify the assumption, by finding f, then we can use the thm.

If $\vec{F} = \nabla f$, then

$$f_x(x,y) = 2y^{3/2}$$

$$f_y(x,y) = 3x\sqrt{y}.$$

$$\vdots$$

$$f(x,y) = 2xy^{3/2} + k \quad (\text{let } k = 0)$$

$$w = \int_c \vec{F} \cdot d\vec{r} = f(2,4) - f(1,1).$$

In 16.1
we learned $\vec{F}$ is a conservative vector field if $\exists f$ s.t. $\vec{F} = \nabla f$. f is called the potential fct for $\vec{F}$.

Conservation of energy (tie in to physics).

How much work to move a particle along $r(t)$, $a \le t \le b$ (call the endpts A & B). thru the force field $\vec{F}$.

Newton's 2nd law of motion. $\vec{F} = m\vec{a}$.

In a force field along C: $\vec{F}(\vec{r}(t)) = m \cdot \vec{r}''(t)$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b m\, \vec{r}\,''(t) \cdot \vec{r}\,'(t)\, dt$$

$$= m \int_a^b \frac{1}{2} \frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) dt$$

note: $\frac{d}{dt}(\vec{r}\,'(t) \cdot \vec{r}\,'(t))$

$= 2\, \vec{r}\,''(t) \cdot \vec{r}\,'(t)$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |\vec{r}\,'(t)|^2\, dt$$

$$= \frac{m}{2} \Big[|\vec{r}\,'(t)|^2 \Big]_a^b$$

$$= \frac{m}{2} \left(\underbrace{|\vec{r}\,'(b)|^2} - \underbrace{|\vec{r}\,'(a)|^2} \right)$$

$\vec{r}\,' = \vec{v}$

$$= \frac{m}{2} \left(|\vec{v}(b)|^2 - |\vec{v}(a)|^2 \right)$$

velocity at points B & A

recall from physics: KE : $k = \frac{1}{2} m v^2$

$\Rightarrow \quad w = k(B) - k(A)$ (Now to bring in PE)

suppose $\vec{F}$ is conservative : $\vec{F} = \vec{\nabla} f$, for some f.

In physics, we define PE : $P = -f$ (why).

$\Rightarrow \vec{\nabla} P = -\vec{\nabla} f$ (this makes sense)

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$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= -\int_C \vec{\nabla} P \cdot d\vec{r}$$

$$= -\left(P(\vec{r}(b)) - P(\vec{r}(a)) \right)$$

$$= P(A) - P(B)$$

$$\Rightarrow \quad k(B) - k(A) = W = P(A) - P(B)$$

OR $$\underbrace{P(A) + k(A)}_{\text{potential + kinetic energy @ Pt A}} = P(B) + k(B)$$

potential + kinetic energy @ Pt A = PE + kE @ pt B.

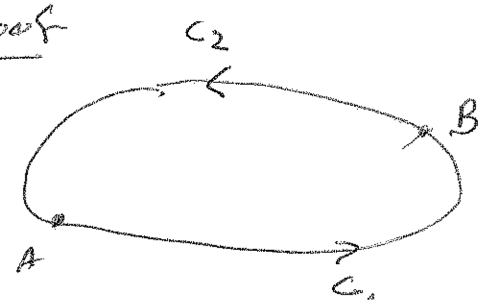
~~recall: $\vec{F}$ is a conservative vector field if $\exists f$ s.t. $\vec{F} = \nabla f$. f is called the potential fct for $\vec{F}$.~~

Dfn: $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 & C_2 in the domain D of $\vec{F}$.

Dfn: closed path (curve)

Thm! $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D iff $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D.

☐ proof



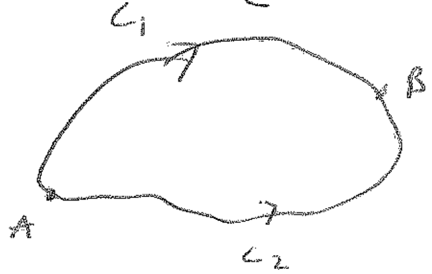
main pt: structure of an "iff" proof.

suppose C is any closed path in D & $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

$(\Rightarrow)$
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$
$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r}$$
$$= 0$$

suppose C is a closed path in D
and $\int_C \vec{F}\cdot d\vec{r} = 0$

$(\Leftarrow)$ $0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r}$



$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}$

$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r}$ is independent of path. $\boxtimes$

<u>Thm</u>: Suppose $\vec{F}$ is a vector field that is cont. on an <u>open</u> <u>connected</u> region D. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a fct f s.t. $\nabla f = \vec{F}$

see proof for notes... open, connected, FToC 1.

<u>Q</u>: How do we determine if a field is conservative?

<u>Thm</u>: If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field where P & Q have cont. first-order partials on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{(by Clairaut's Thm).}$$

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> **4 THEOREM** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

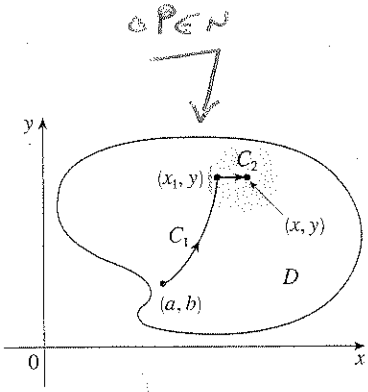


FIGURE 4

PROOF Let $A(a, b)$ be a fixed point in D. We construct the desired potential function f by defining

$$f(x, y) = \int_{(a, b)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point (x, y) in D. Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$. Since D is open, there exists a disk contained in D with center (x, y). Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y). (See Figure 4.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \overbrace{\int_{(a, b)}^{(x_1, y)} \mathbf{F} \cdot d\mathbf{r}}^{\text{fct of } y} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

note: comment on dfns. of connected & independence of path.

If we write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$$

On C_2, y is constant, so $dy = 0$. Using t as the parameter, where $x_1 \leq t \leq x$, we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \overbrace{\frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt}^{\text{FToC 1}} = P(x, y)$$

by Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). A similar argument, using a vertical line segment (see Figure 5), shows that

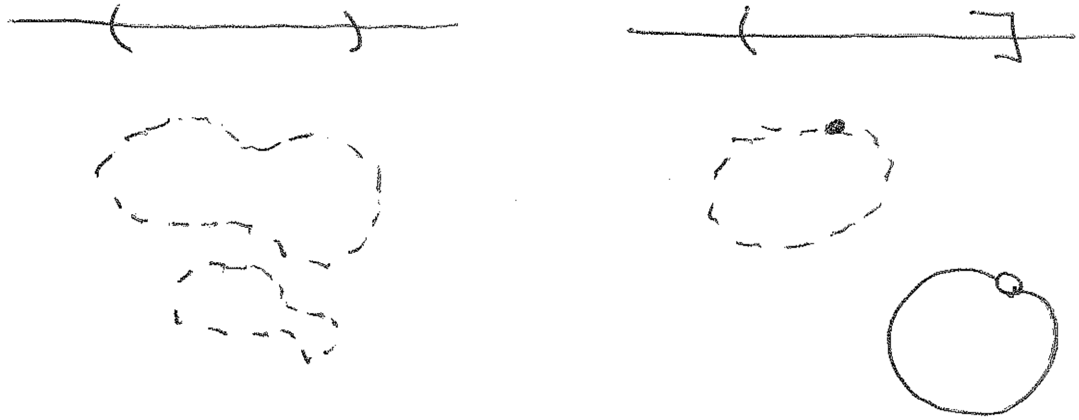
$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y)$$

Thus
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f$$

which says that $\mathbf{F}$ is conservative. □

YES NO

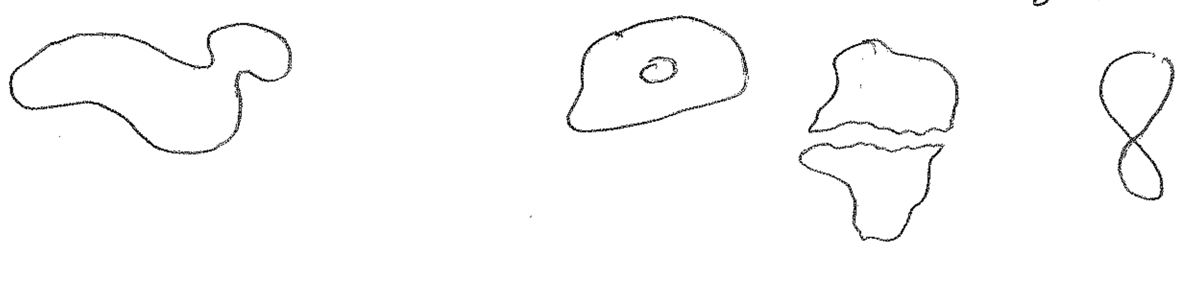
<u>OPEN</u> (every pt in D can be surrounded by an open disk (centered @ the pt) entirely in D.)



<u>connected.</u> (any 2 pts in D can be connected by a path in D)

<u>simple curves</u> (no intersections)

<u>simply connected regions</u> (all simple closed curves in D contain only pts in D).



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Do we have an answer to the preceding question ... no, we still don't know how to determine if a field is conservative.

The following definition refers to a simply-connected region (see graphs)

<u>Thm:</u> Let $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field on an open simply-connected region D. Suppose that P & Q have continuous 1st order partial derivatives and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout D (Clairaut's Thm). Then $\vec{F}$ is conservative.

The proof is sketched out in 16.4.

<u>Key:</u> If $\vec{F}$ is conservative, $W = \int_C \vec{F} \cdot d\vec{r}$ can be calculated w/ the Fundamental Thm for Line Integrals.