15.10: Change of Variables.

This section is a multivariate approach to integration by substitution.

ex: $\int_a^b \sin 2x \, dx = \int_{2a}^{2b} \sin(u) \frac{1}{2} \, du$

$\uparrow$

scaling factor.

Let's start a few examples to better see the need and sticky point

ex1: $I = \iint_R \sin(9x^2+4y^2)\, dA$ where R is the region in QI bounded by $9x^2+4y^2=1$.

Let $u = 3x$ and $v = 2y$

$\Rightarrow I = \iint_{\substack{\text{unit}\\ \text{disk}}} \sin(u^2+v^2) \begin{pmatrix}\text{scaling}\\ \text{factor}\end{pmatrix} dA'$ ← in terms of u & v.

ex2: $I = \iint_R \frac{x-2y}{3x-y}\, dA$ where R is the parallelogram enclosed by $x-2y=0$; $x-2y=4$
$3x-y=1$; $3x-y=8$

Let $u = x-2y$ and $v = 3x-y$

$I = \int_1^8 \int_0^4 \frac{u}{v} \begin{pmatrix}\text{scaling}\\ \text{factor}\end{pmatrix} du\, dv.$

ex3: $I = \iint_R (x+y)e^{x^2-y^2}\, dA$ where R is the rectangle enclosed by $x-y=0$; $x-y=2$
$x+y=0$; $x+y=3$

Let $u = x-y$ and $v = x+y$

$$\Rightarrow \quad I = \int_0^3 \int_0^2 v e^{uv} \begin{pmatrix} \text{scaling} \\ \text{factor} \end{pmatrix} du\, dv$$

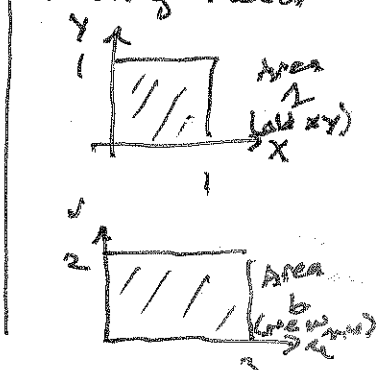
ex1 rev: $I = \iint_R \sin(9x^2+4y^2)\, dA$ where R is the region in Q1 bounded by $9x^2+4y^2=1$.

Let $u = 3x$ and $v = 2y$

$\Rightarrow \quad x = \frac{u}{3}$ and $y = \frac{v}{2}$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 0 \\ 0 & 1/2 \end{vmatrix} = \frac{1}{6}$$

Find geometric scaling factor



$$\Rightarrow I = \frac{1}{6}\iint_D \sin(u^2+v^2)\, dA'$$

where A' is the unit disk in Q1 for (u,v).

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}}\int_0^1 \sin(r^2)\, r\, dr\, d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}\cos(r^2)\right]_0^1 d\theta$$

$$= -\frac{1}{12}\int_0^{\pi/2} (\cos(1) - 1)\, d\theta$$

$$= \frac{\pi}{24}(1-\cos(1))$$

Scaling factor $= \dfrac{\text{old}}{\text{new}} = \dfrac{1}{6}$

15.10
3.19

$$I = \iint_R \frac{x-2y}{3x-y}\, dA$$ where R is the parallelogram enclosed by $x-2y=0$; $x-2y=4$

$3x-y=1$; $3x-y=8$

Let $u = x-2y$ and $v = 3x-y$

$$I = \int_1^8 \int_0^4 \frac{u}{v} \left(\text{scaling factor}\right) du\, dv$$

$$= \frac{1}{5}\int_1^8 \left[\frac{1}{2}\frac{u^2}{v}\right]_0^4 dv$$

$$= \frac{1}{10}\int_1^8 \frac{16}{v}\, dv$$

$$= \frac{8}{5}\Big[\ln|v|\Big]_1^8$$

$$= \frac{8}{5}\ln 8.$$

The scaling factor.

(I) $$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$\det(A) = 5$

so the (u,v) rectangle is 5 times larger than the (x,y) parallelogram.

(II) $x = -\frac{1}{5}u + \frac{2}{5}v$

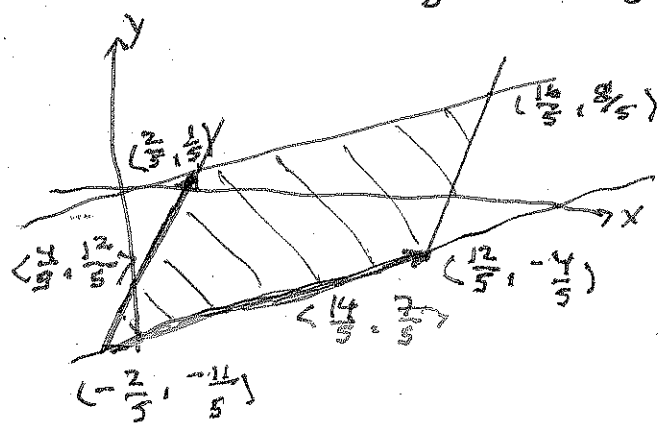
$y = -\frac{3}{5}u + \frac{1}{5}v$

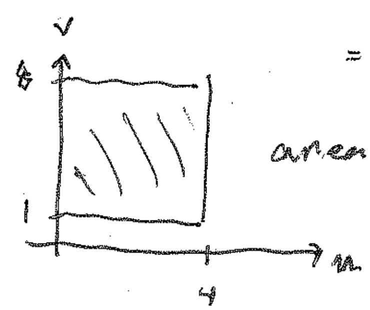
$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix}$$

$$= -\frac{1}{25} + \frac{6}{25}$$

$$= \frac{1}{5}$$

Geometric Scaling Factors





area $= 28$ (new uv)

area: $$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 14/5 & 7/5 & 0 \\ 4/5 & 12/5 & 0 \end{vmatrix} = \langle 0, 0, \frac{140}{25} \rangle$$

area $= \frac{28}{5}$ (old xy)

Scaling factor $= \frac{\text{old}}{\text{new}} = \frac{28/5}{28} = \frac{1}{5}$

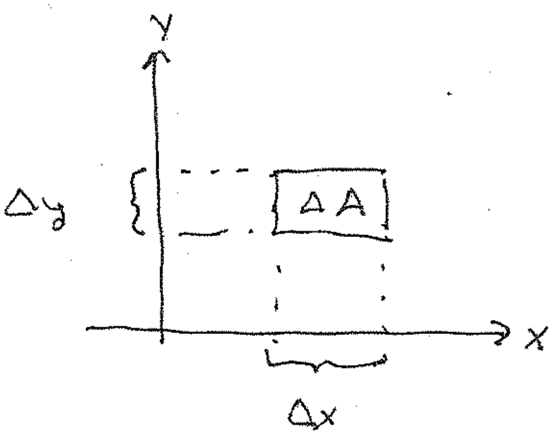
15.10
4/9

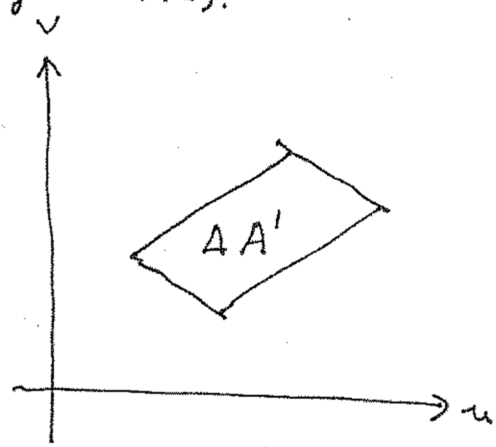
Scaling Factor Derivation (Linear)

For a substitution $u = u(x,y)$ and $v = v(x,y)$, what is the scaling factor $dx\,dy$ vs. $du\,dv$?

Suppose $u = 3x - 2y$ (to simplify the integrand on the bounds).
$v = x + y$.

Find the relation between $dA = dx\,dy$ & $dA' = du\,dv$. (exchange rate).





a parallelogram since <u>linear trans</u> *

we should have a constant scaling factor that is independent of the choice of rect.

so let's use the unit square.

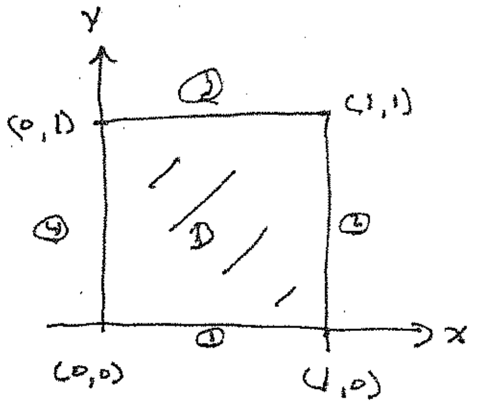
$$* \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ x + y \end{bmatrix}$$

$$= x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

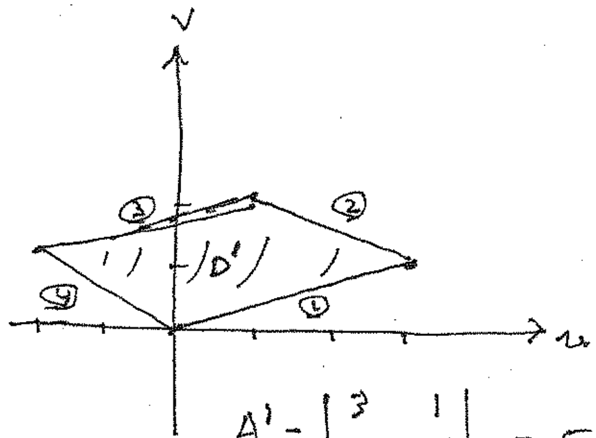
$$= \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$= A\vec{x}$ so it's a linear transformation.

$\boxed{\begin{matrix} 15.10 \\ 5/19 \end{matrix}}$



$A = 1$



$A' = \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5$

For any rectangle, ΔA is $\frac{1}{5}\,\Delta A' \Rightarrow dA' = 5\,dA$

$du\,dv = 5\,dx\,dy$

Q: why does the determinant give the area?

↳ $\iint_D f\,dx\,dy = \iint_{D'} f'\,\frac{1}{5}\,du\,dv$

in terms of x & y — in terms of u & v

it isn't always a const. scaling factor because the transformation $T: (x,y) \to (u,v)$ isn't necessarily linear But we can address this thru linear approximation.

$I = \iint_R (x+y)e^{x^2-y^2}\,dA$ where R is the

rectangle enclosed by $x-y=0$; $x-y=2$

$x+y=0$; $x+y=3$

Let $u = x-y$ and $v = x+y$

$$\Rightarrow \quad I = \int_0^3\int_0^2 v e^{uv}\left(\begin{matrix}\text{scaling}\\ \text{factor}\end{matrix}\right) du\,dv$$

$$\begin{tikzcd} (x,y) \arrow[r, bend left, "\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}"] & (u,v) \arrow[l, bend left, "\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}"] \end{tikzcd}$$

scaling factor of $\frac{1}{2}$

$$I = \int_0^3\int_0^2 v e^{uv}\tfrac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_0^3 \left[e^{uv}\right]_0^2 dv$$

$$= \frac{1}{2}\int_0^3 (e^{2v}-1)\,dv$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2v} - v\right]_0^3$$

$$= \frac{1}{2}\left(\frac{1}{2}e^6 - 3 - \frac{1}{2} + 0\right)$$

$$= \frac{1}{4}e^6 - \frac{7}{4}$$

Note: If you switch u & v ... that is $u = x+y$

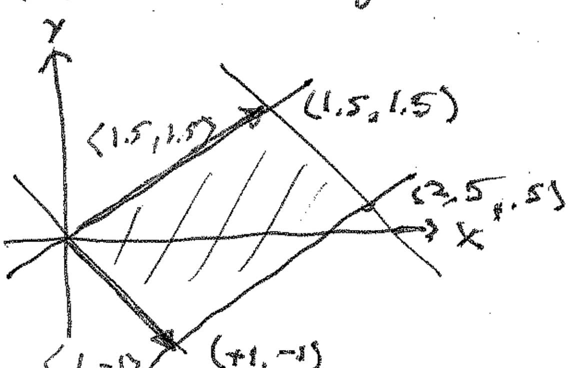
$v = x-y$

we find the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}.$$

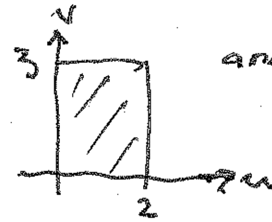
but the scaling factor is the same $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

Geometric Scaling Factor



Area: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1.5 & 1.5 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \langle 0, 0, -3\rangle$

area = 3 (old $x\,y$)



area = 6 (new $u\,v$)

$$\text{scaling factor} = \frac{\text{old}}{\text{new}} = \frac{3}{6} = \frac{1}{2}$$

General case: (differentials).

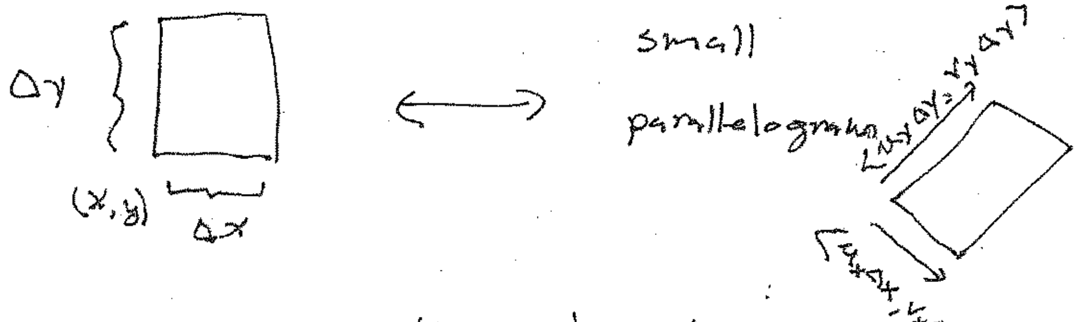
15.10 2/9

$u = u(x,y) \qquad \Delta u \approx u_x \Delta x + u_y \Delta y$

$v = v(x,y) \qquad \Delta v \approx v_x \Delta x + v_y \Delta y$

$$\text{or} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

So a small rect. in (x,y) has the image of a small parallelogram under T



same argument ... the determinant tells us how we scaled coords.

$\langle \Delta x, 0 \rangle \rightsquigarrow \langle \Delta u, \Delta v \rangle \approx \langle u_x \Delta x, v_x \Delta x \rangle$

sides of par.

$\langle 0, \Delta y \rangle \rightsquigarrow \langle \Delta u, \Delta v \rangle \approx \langle u_y \Delta y, v_y \Delta y \rangle$

the area = det( ) dx dy.

Jacobian: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Then $du\,dv = \left| \dfrac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$

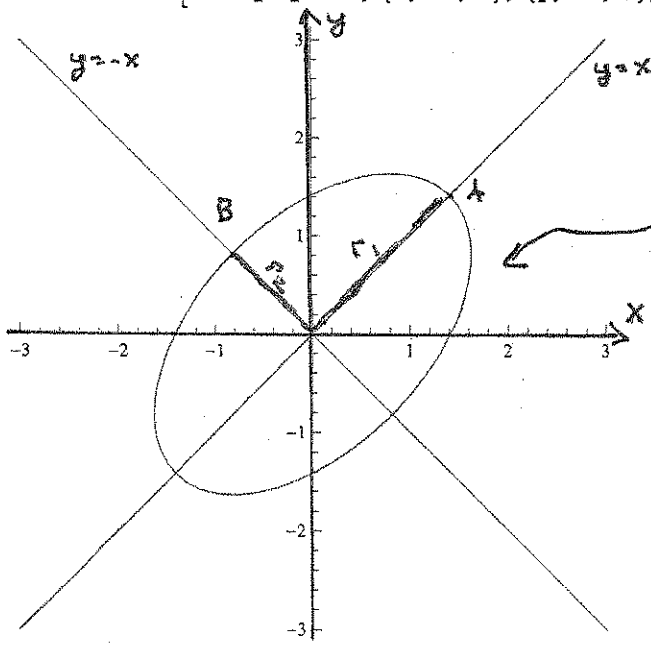
HARD EXAMPLE

$I = \iint_R x^2 - xy + y^2 \, dA$ where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$

$\boxed{\begin{matrix} 15.10 \\ 8/9 \end{matrix}}$

```
In[13]:= Show[{Plot[{x, -x}, {x, -3, 3}],
   ContourPlot[x^2 - x y + y^2 == 2, {x, -3, 3}, {y, -3, 3}, Axes -> True]}, AspectRatio -> 1]
```

Out[13]=



$x^2 - xy + y^2 = 2$

It's an ellipse rotated by $\frac{\pi}{4}$ c.c.w and scaled.

How is it scaled?

Find (A). $x^2 - x^2 + x^2 = 2$ (since $y = x$)

$\Rightarrow x = \pm\sqrt{2}$

$\Rightarrow A(\sqrt{2}, \sqrt{2}) \longrightarrow$ major axis has length $2 \cdot 2$

$r_1 = 2$

Find (B). $x^2 + x^2 + x^2 = 2$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$

$\Rightarrow B\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \longrightarrow$ minor axis has length $2 \cdot \frac{2}{\sqrt{3}}$

$r_2 = \frac{2}{\sqrt{3}}$

15.10
9/9

our new variables (u, v) should be s.t.

$$x^2 - xy + y^2 = 2 \longrightarrow u^2 + v^2 = 1$$

$$(u, v) = \begin{matrix}\text{compress} \\ x \text{ by } \tfrac{1}{2} \\ \text{stretch} \\ y \text{ by } \sqrt{3}/2\end{matrix} \longleftarrow \begin{matrix}\text{rotate} \\ \pi/4 \\ \text{c.w}\end{matrix} \longleftarrow (x, y)$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow u = \frac{1}{2\sqrt{2}}(x+y) \quad \text{and} \quad v = \frac{\sqrt{3}}{2\sqrt{2}}(y-x)$$

and $x = \sqrt{2}\, u - \sqrt{\frac{2}{3}}\, v$ and $y = \sqrt{2}\, u + \sqrt{\frac{2}{3}}\, v$

Integrand: $x^2 - xy + y^2 = 2(u^2 + v^2)$

Region: $x^2 - xy + y^2 \leq 2 \Rightarrow u^2 + v^2 \leq 1$

Scaling factor: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \end{vmatrix} = \frac{4}{\sqrt{3}}$

$$\Rightarrow I = \iint_{u^2+v^2 \leq 1} 2(u^2 + v^2) \frac{4}{\sqrt{3}} \, du\, dv$$

$$= \int_0^{2\pi} \int_0^1 2r^2 \cdot \frac{4}{\sqrt{3}} r \, dr\, d\theta$$

$$= 2\pi \cdot \frac{4}{\sqrt{3}} \cdot 2 \cdot \frac{1}{4}$$

$$= \frac{4\pi}{\sqrt{3}}$$