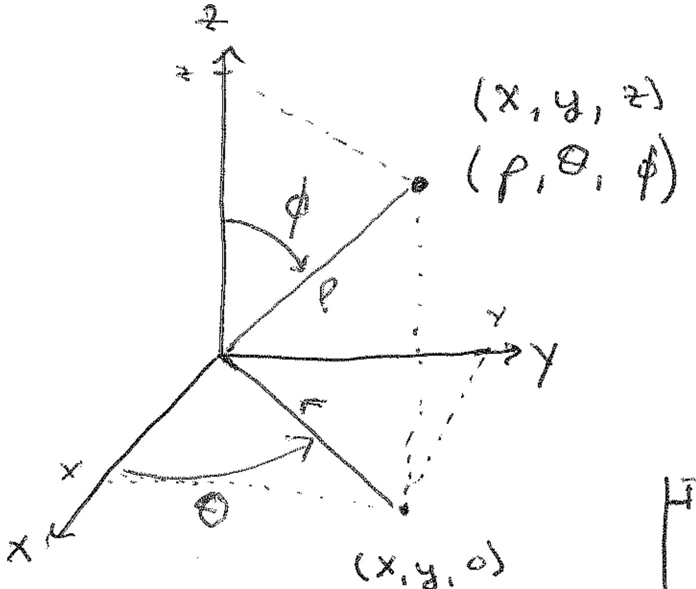
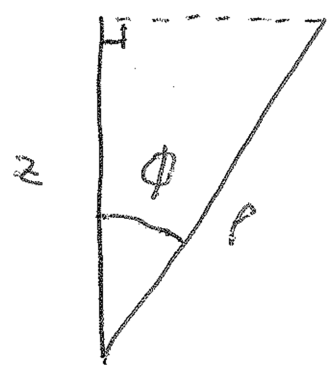
15.9: Triple Integrals w/ Spherical Coordinates.



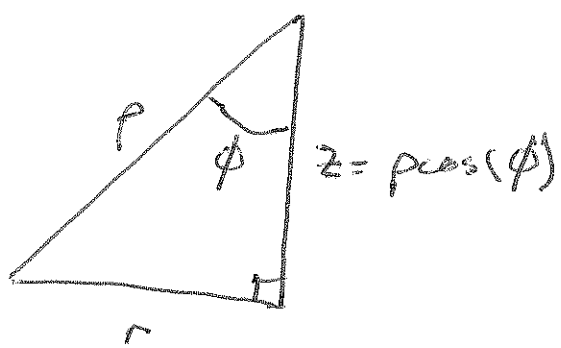
what is the relationship between (x, y, z) & (ρ, θ, ϕ)

$\rho \geq 0$ & $0 \leq \phi \leq \pi$.



$$\cos(\phi) = \frac{z}{\rho}$$

OR $z = \rho \cos(\phi)$



$$\cos(\phi) = \frac{z}{\rho}$$

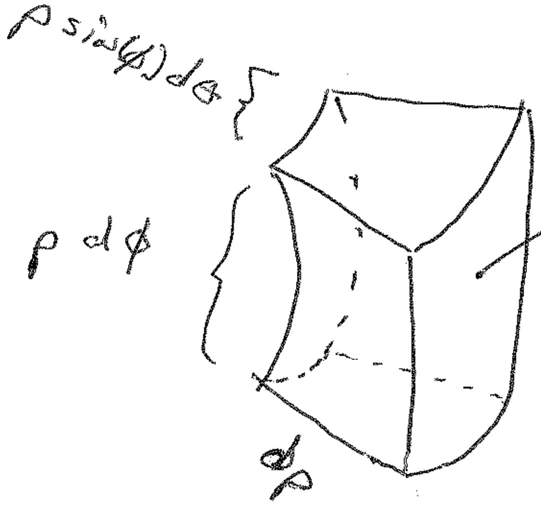
$$\sin(\phi) = \frac{r}{\rho}$$

OR $r = \rho \sin(\phi)$

AND
$$x = r\cos(\theta) \qquad \Longrightarrow \qquad x = \rho \sin(\phi)\cos(\theta)$$
$$y = r\sin(\theta) \qquad\qquad\quad y = \rho \sin(\phi)\sin(\theta)$$

AND $\rho^2 = x^2 + y^2 + z^2$

The differential

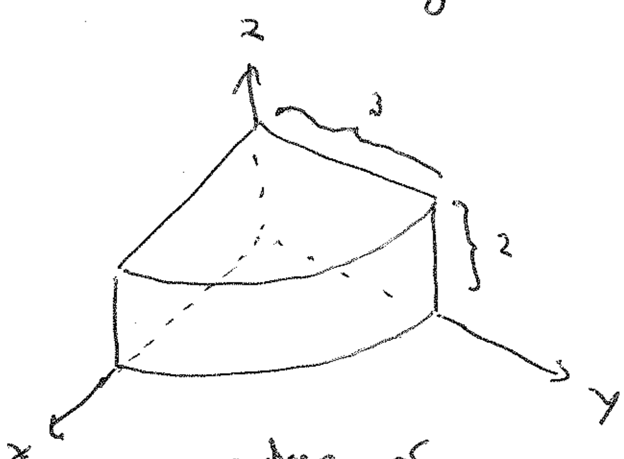


$dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$

see cool mathematica graphic

<u>Question</u>: what do (a) $\rho = c$, (b) $\theta = c$, & (c) $\phi = c$ look like?

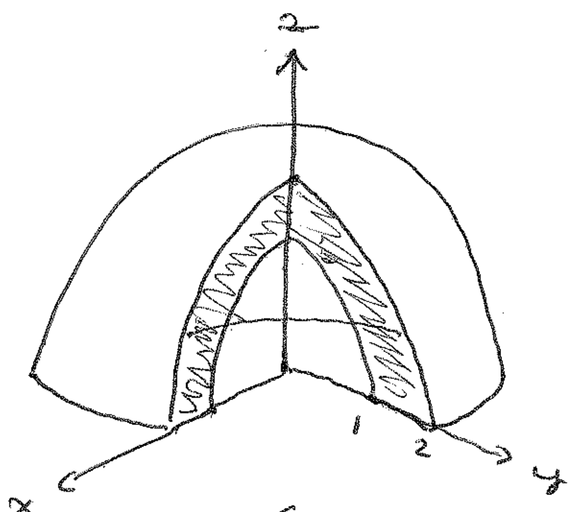
<u>Ex1</u>: (see pics) Set up integrals over the regions



wedge of cheese

cylindrical

$$\int_0^{\frac{\pi}{2}}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$



top of a sphere w/ a qtr cut out.

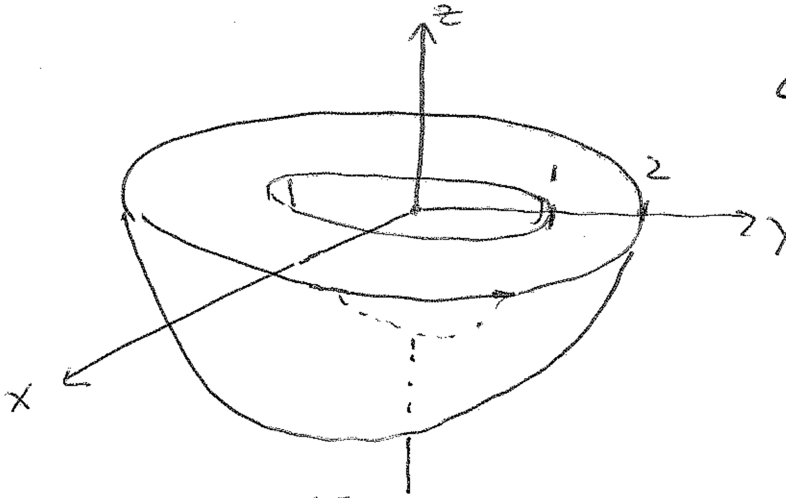
spherical

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{2\pi}\int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Ex2: Sketch the solid whose volume is given by the integral

$$\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_1^2 1\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

& calculate/eval the integral



cantaloupe w/ the seeds removed.

$$V = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\left[\frac{\rho^3}{3}\sin\phi\right]_1^2 d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{8-1}{3}\sin\phi\, d\phi\, d\theta$$

$$= \frac{7}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_{\frac{\pi}{2}}^{\pi} d\theta$$

$$= \frac{7}{3}\int_0^{2\pi}\big(-(-1) - 0\big)\, d\theta$$

$$= \frac{14}{3}\pi$$

check

$$V = \frac{1}{2}\left(\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3\right)$$

$$= \frac{1}{2}\left(\frac{32\pi}{3} - \frac{4\pi}{3}\right)$$

$$= \frac{28\pi}{6}$$

$$= \frac{14\pi}{3}$$

Ex3: evaluate $\iiint_S xyz\,dV$ where S is bounded between the spheres $\rho=4$ $\rho=2$ & above the cone $\phi=\frac{\pi}{3}$

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_2^4 \rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta\,\rho\cos\phi\,\rho^2\sin\phi\;d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_2^4 \rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\underbrace{2\sin\theta\cos\theta}_{\sin 2\theta}\,d\theta \int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi \int_2^4 \rho^5\,d\rho$$

$$= \left[-\frac{\cos 2\theta}{4}\right]_0^{2\pi} \left[\frac{\sin^4\phi}{4}\right]_0^{\pi/3} \cdot \left[\frac{\rho^6}{6}\right]_2^4$$

$$= \left(-\frac{1}{4}-\left(-\frac{1}{4}\right)\right)\cdot \sim \cdot \sim$$

$$= 0$$

Find the scaling factor

$x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin(\phi)\sin(\theta) & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos(\phi) & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$- \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= -\rho^2\cos\phi\left(\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta)\right)$$

$$- \rho^2\sin^3\phi\,(\cos^2\theta + \sin^2\theta)$$

$$= -\rho^2\sin\phi\,(\cos^2\phi + \sin^2\phi)$$

$$= -\rho^2\sin\phi$$

The scaling factor is the <u>magnitude</u> of the Jacobian.

$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$