

15.5: Apps of Double Integrals

(I) Physics

- mass
- center of mass
- radius of inertia
- moments
- moments of inertia

(II) Probability: Joint density fcts of 2 variables.

If $p(x,y)$ gives the density of a lamina at a pt (x,y) , why does $p(x,y)dA$ give the mass of the infinitesimal rectangular region?

$$\text{Mass: } m = \iint_D p(x,y) dA$$

ex1: Find the following of a lamina w/ uniform density 1 enclosed by the cardioid $r = 1 + \cos \theta$.

(a) The mass

$$m = 2 \int_0^\pi \int_0^{1+\cos \theta} 1 \cdot r \, dr \, d\theta$$

$$= \frac{3}{2} \pi \quad (\text{From 15.4})$$

The moment of a particle is the product of its mass & dist. from the axis.

$$m_x = \iint_D y \underbrace{\rho(x,y)}_{\text{mass}} dA \quad \& \quad m_y = \iint_D x \underbrace{\rho(x,y)}_{\text{mass}} dA$$

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(b) Find the moments.

$$\begin{aligned}
 m_x &= \int_0^{2\pi} \int_0^{1+\cos\theta} y \cdot 1 \cdot r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 \sin\theta \, dr \, d\theta \\
 &= \int_0^{2\pi} \frac{1}{3} \left[r^3 \sin\theta \right]_0^{1+\cos\theta} d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} (1+\cos\theta)^3 \sin\theta \, d\theta \\
 &= -\frac{1}{3} \int_2^2 u^3 \, du \\
 &= 0
 \end{aligned}$$

Let $u = 1 + \cos\theta$
 $du = -\sin\theta \, d\theta$
 $u(0) = 2$
 $u(2\pi) = 2$

$$\begin{aligned}
 m_y &= \int_0^{2\pi} \int_0^{1+\cos\theta} x \cdot 1 \cdot r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 \cos\theta \, dr \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \left[r^3 \cos\theta \right]_0^{1+\cos\theta} d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} (1+\cos\theta)^3 \cos\theta \, d\theta \\
 &= \frac{5\pi}{4} \quad (\text{via Mathematica})
 \end{aligned}$$

The center of mass is at $\bar{x} = \frac{M_y}{m}$ & $\bar{y} = \frac{M_x}{m}$

(c) Find the centroid.

$$\bar{x} = \frac{5\pi}{4} \cdot \frac{2}{3\pi} = \frac{5}{6} \quad \& \quad \bar{y} = 0$$

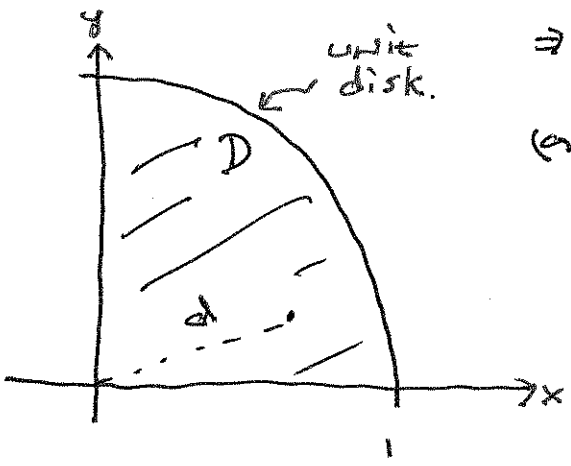
$$\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{5}{6}, 0\right)$$

see pic on page 4.

Note that \bar{y} is obvious given the graph. //

After ex 2 { The moment of inertia are calculated in a similar manner.
 $I_x = \iint_D y^2 \rho(x,y) dA$ and $I_y = \iint_D x^2 \rho(x,y) dA$.

ex 2: A lamina occupies the part of the unit circle in Q1. The density @ any pt is proportional to the square of the dist. from the origin.



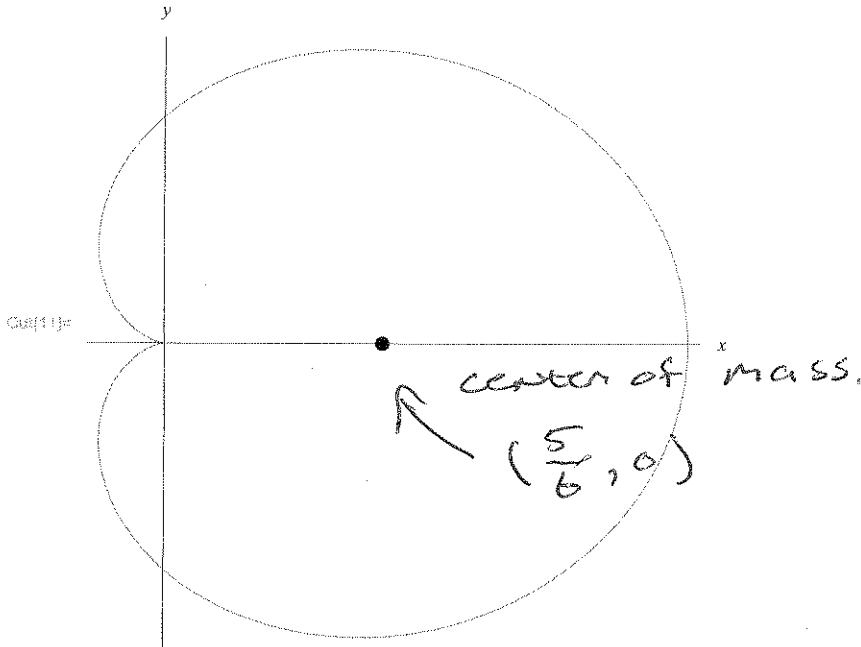
$$\Rightarrow \rho(x,y) = k(x^2 + y^2)$$

$$\begin{aligned} (a) \quad m &= k \iint_D (x^2 + y^2) dA \\ &= k \int_0^{\pi/2} \int_0^1 r^2 \cdot r \, dr \, d\theta \\ &= k \frac{1}{4} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{8} k \end{aligned}$$

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Show[PolarPlot[1 + Cos[θ], {θ, 0, 2 π}, AspectRatio → 1],
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Graphics[{{PointSize → Large, Point[{ $\frac{5}{6}$ , 0}]}}, Ticks → None, AxesLabel → {x, y}]
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note: we could determine $\bar{y} = 0$ in this case by symmetry. But, if ρ had been $\rho = \text{constant}$, then ... we would have had to work.

$$\begin{aligned}
 (b) \quad m_x &= \iint_D y(x^2 + y^2) \, dA \\
 &= \int_0^{\pi/2} \int_0^1 r \sin \theta \cdot r^2 \cdot r \, dr \, d\theta \\
 &= \int_0^{\pi/2} \sin \theta \, d\theta \cdot \int_0^1 r^4 \, dr \\
 &= \frac{1}{5}
 \end{aligned}$$

$$m_y = \frac{1}{5} \quad (\text{by symmetry})$$

$$\text{so } (\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi} \right)$$

↳ moment of inertia formulas.

$$\begin{aligned}
 (c) \quad I_x &= \iint_D y^2 (x^2 + y^2) \, dA \\
 &= \int_0^{\pi/2} \sin^3 \theta \, d\theta \cdot \int_0^1 r^5 \, dr
 \end{aligned}$$

$$(d) \quad I_y = \iint_D x^2 (x^2 + y^2) \, dA$$

$$(e) \quad I_o = I_x + I_y \quad (\text{the moment about the origin})$$

The moment of inertia is to rotation what mass is to linear motion.

we will skip the radius of gyration.

Probability

The joint density fct of the random variables X & Y is a fct f of two variables s.t. the probability that (X, Y) lies in D is:

$$Pr((X, Y) \in D) = \iint_D f(x, y) dA.$$

Note: $\iint_{\mathbb{R}^2} f(x, y) dA = 1$... why?

ex3: Suppose X & Y are random variables w/ joint density fct.

$$f(x, y) = \begin{cases} 0.1 e^{-(0.5x + 0.2y)} & , x, y \geq 0 \\ 0 & , \text{else} \end{cases}$$

- a) verify f is a joint density fct.
- b) find $Pr(X \leq 2 \text{ and } Y \leq 4)$
- c) find $Pr(Y \geq 1)$
- d) Find the expected value of X :

$$\mu_1 = \iint_{\mathbb{R}^2} x \cdot f(x, y) dA$$

$$\begin{aligned}
 (a) \iint_{\mathbb{R}^2} f(x, y) dA &= \int_0^{\infty} \int_0^{\infty} 0.1 e^{-(0.5x + 0.2y)} dx dy \\
 &= 0.1 \int_0^{\infty} e^{-0.5x} dx \int_0^{\infty} e^{-0.2y} dy \\
 &= 0.1 \left[-2e^{-0.5x} \right]_0^{\infty} \left[-5e^{-0.2y} \right]_0^{\infty} \\
 &= 1
 \end{aligned}$$

$$(b) \Pr(X \leq 2 \text{ and } Y \leq 4)$$

$$\begin{aligned}
 &= 0.1 \left[-2e^{-0.5x} \right]_0^2 \left[-5e^{-0.2y} \right]_0^4 \\
 &= (e^{-1} - 1)(e^{-0.8} - 1) \\
 &= 0.348
 \end{aligned}$$

$$(c) \Pr(Y \geq 1) = 1 - \Pr(Y \leq 1)$$

$$\begin{aligned}
 &= 1 - \left[e^{-0.5x} \right]_0^{\infty} \left[e^{-0.2y} \right]_0^1 \\
 &= 1 - (0 - 1)(e^{-0.2} - 1) \\
 &= 0.819
 \end{aligned}$$

(d) Find the expected value of X :

$$\begin{aligned}
 \mu_1 &= \iint_{\mathbb{R}^2} x f(x, y) dA \\
 &= \int_0^{\infty} \int_0^{\infty} x \cdot 0.1 e^{-(0.5x + 0.2y)} dA \\
 &= +0.1 \left[\frac{1 + 0.5x}{0.5^2} e^{-0.5x} \right]_0^{\infty} \left[-5 e^{-0.2y} \right]_0^{\infty} \\
 &= 0.1 \frac{1}{0.5^2} + 5 \\
 &= 2
 \end{aligned}$$

Other notes:

Formulas for the second moments,

Moment of inertia about:

the x-axis: $I_x = \iint_D y^2 \rho(x,y) dA$

the y-axis: $I_y = \iint_D x^2 \rho(x,y) dA$

the pole: $I_o = I_x + I_y$

The radius of gyration w.r.t.:

the x-axis: $\bar{y} = \frac{I_x}{m}$

the y-axis: $\bar{x} = \frac{I_y}{m}$

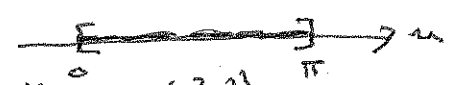
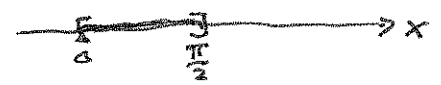
(\bar{x}, \bar{y}) is the pt where the mass could be concentrated w/o changing the moment of inertia.

Scaling Factors

$$\int_0^{\pi/2} \sin(2x) dx = \int_0^{\pi} \sin u \cdot \frac{1}{2} du = 1$$

Let $u = 2x$
 $du = 2dx$

↑
scaling factor.

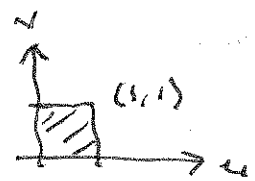
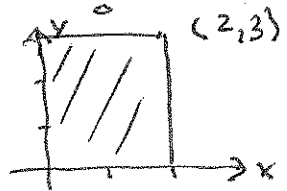


$$\int_0^{\pi/2} \int_0^{\pi/3} \sin(2x) \sin(3y) dy dx$$

Let $u = 2x, v = 3y$
 $du = 2dx, dv = 3dy$

$$= \int_0^{\pi} \int_0^{\pi} \sin(u) \sin(v) \frac{1}{2} \cdot \frac{1}{3} du dv$$

$$= 2 \cdot 2 \cdot \frac{1}{6} = \frac{2}{3}$$



In 15.10, we will learn about using the Jacobian for change of variables.

$$\text{Jacobian } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

polar to rectangular.