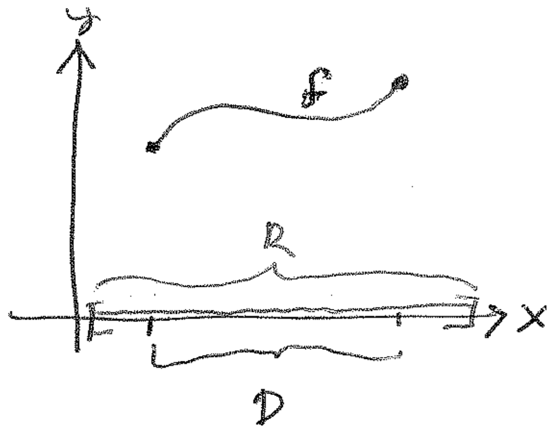
15.3 : Double Integrals over General Regions.

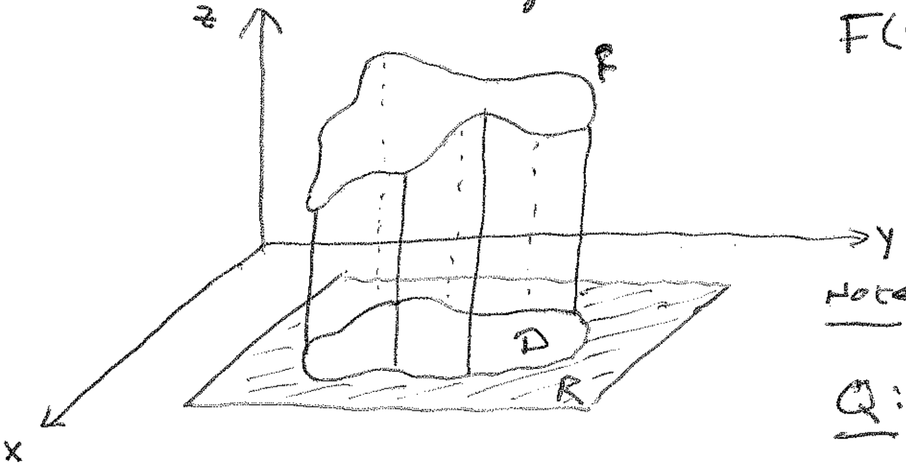
single integrals



$$F(x) = \begin{cases} f(x), & x \in D \\ 0, & x \in R \text{ AND } x \notin D. \end{cases}$$

Note: D is bounded (otherwise we have an improper integral).

double integrals



$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \text{ and } (x,y) \notin D \end{cases}$$

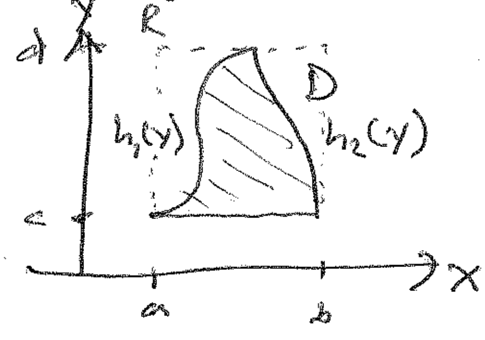
Note: D is bounded

Q: Is F cont?

☆ Fubini's Thm Applies.

If $D = \{(x,y) \mid c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$

That is:



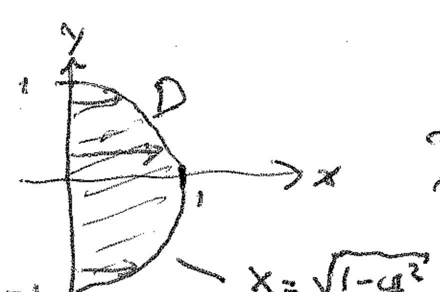
$$\iint_D F(x,y)\,dA = \iint_R F(x,y)\,dA$$
$$= \int_c^d \int_a^b F(x,y)\,dx\,dy$$
$$= \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} F(x,y)\,dx\,dy$$
$$= \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy.$$

Note: the text refers to regions D of this sort as Type II regions.

Note: A double integral is a number associated w/ a fcn f & a region D, and this number exists & has a meaning independent of an particular method of computing it.

An iterated integral is a double integral plus a built in computational procedure.

Ex 1: $\iint_D xy^2\,dA$ where D is enclosed by $x=0$ & $x=\sqrt{1-y^2}$



$D = \{(x,y) \mid -1 \le y \le 1$ AND $0 \le x \le \sqrt{1-y^2}$

$$\hookrightarrow \iint_D x y^2 \, dA = \int_{-1}^{1} \int_{x=0}^{x=\sqrt{1-y^2}} x y^2 \, dx \, dy.$$

$$= \int_{-1}^{1} \left[\frac{x^2}{2} y^2 \right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1} \frac{1-y^2}{2} \cdot y^2 \, dy$$

$$= \left[\frac{y^3}{6} - \frac{y^5}{10} \right]_{-1}^{1}$$

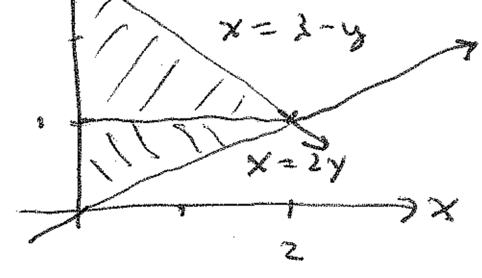
$$= \left(\frac{1}{6} - \frac{1}{10} \right) - \left(-\frac{1}{6} + \frac{1}{10} \right)$$

$$= \frac{1}{3} - \frac{2}{10} = \frac{4}{30} = \frac{2}{15}$$

Ex 2: In evaluating a double integral over D, the following was found.

$$\iint_D f(x,y) \, dA = \int_0^1 \int_0^{2y} f(x,y) \, dx \, dy + \int_1^3 \int_0^{3-y} f(x,y) \, dx \, dy.$$

sketch D & express the double integral w/ reversed order of integration.

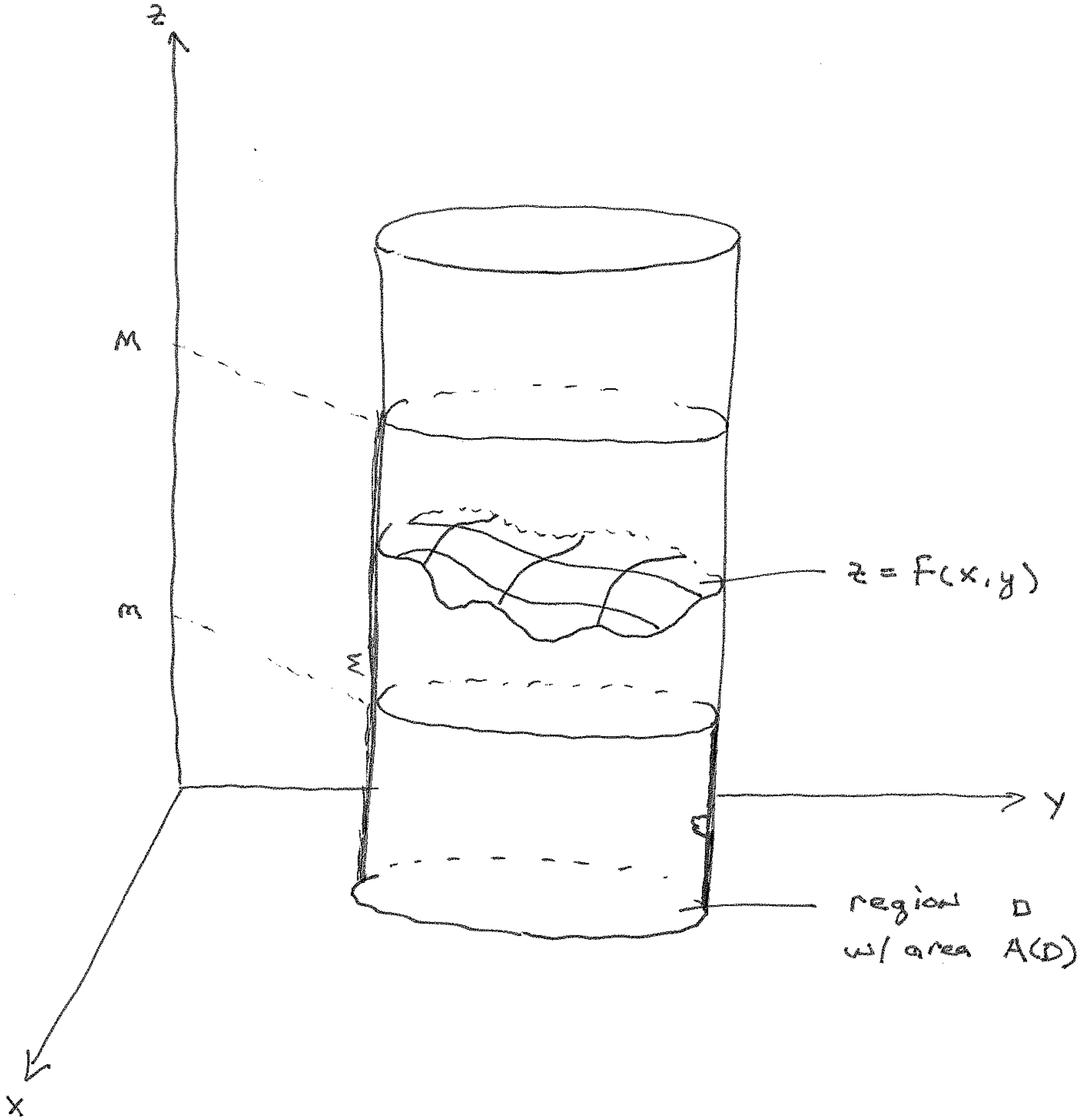


$$\iint_D f(x,y) \, dA = \int_0^2 \int_{y=\frac{1}{2}x}^{y=3-x} f(x,y) \, dy \, dx$$

property of the double integral

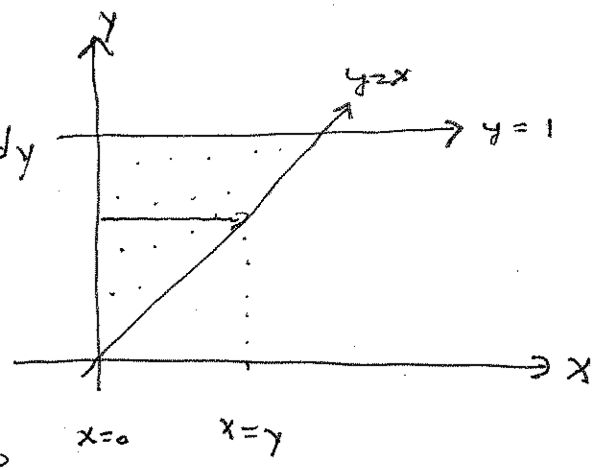
$\boxed{11}$ If $m \le f(x,y) \le M$ in D, then

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D)$$



additional double integrals.

ex 3: $\iint_D x\sqrt{y^2-x^2}\,dA \qquad D=\{(x,y)\mid 0\le y\le 1 \;\&\; 0\le x\le y\}$



$$= \int_0^1 \int_{x=0}^{x=y} x\sqrt{y^2-x^2}\,dx\,dy$$

$$= -\frac{1}{2}\int_0^1 \int_{u=y^2}^{u=0} \sqrt{u}\,du\,dy$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{2}{3}u^{3/2}\right]_{u=y^2}^{u=0} dy \qquad u = y^2 - x^2$$

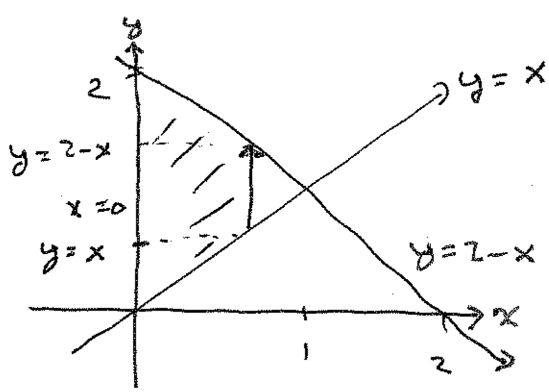
$$= -\frac{1}{2}\cdot\frac{2}{3}\int_0^1 -y^3\,dy \qquad \frac{du}{-2} = x\,dx$$

$$= \frac{1}{3}\cdot\frac{1}{4}\left[y^4\right]_0^1$$

$$= \frac{1}{12}$$

ex4: Find the volume of the region bounded by the planes $z=x$; $y=x$; $x+y=2$; $z=0$.

$$V = \int_0^1 \int_{y=x}^{y=2-x} (x-0)\,dy\,dx.$$



```
function [] = pd(h,option)
% [] = pd(h,option)
% f(x,y) = x^2 + y^3
% h is the step size
% options: 'f', 'fx' and 'fy'

h = h; %step size
[x,y] = meshgrid(-2:h:2,-2:h:2); %domain

z0 = x.^2-y.^3;
zX = (x+h).^2-y.^3;
zY = x.^2-(y+h).^3;

switch option
   case 'f'
      surf(x,y,z0);
   case 'fx'
      surf(x,y,(zX-z0)/h);
   case 'fy'
      surf(x,y,(zY-z0)/h);
   otherwise
      disp('Unknown option.')
end
```