15.2: Iterated Integrals

Overview:

(1) suppose $z = f(x,y)$ is integrable over $R = [a,b] \times [c,d]$.

(2) in $\int_c^d f(x,y)\,dy$ we take x as fixed and f integrated WRT y.

(3) we call (2.) partial integration WRT y. and y is eliminated.

(4) we are left w/ $A(x) = \int_c^d f(x,y)\,dy$

$$\Rightarrow \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

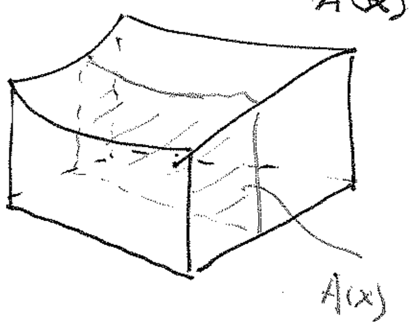
(iterated integral).

our method: work from the inside out.

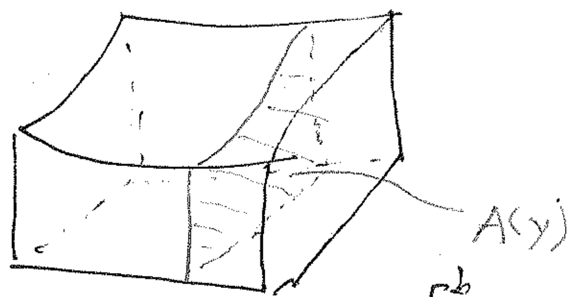
ex1: calculate $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\,dy\,dx$

and $\int_1^2 \int_0^1 (4x^3 - 9x^2y^2)\,dx\,dy$.

Intuitively, the same volume is swept out by $A(x)$ & $A(y)$.



$$\int_c^d f(x,y)\,dy$$



$$\int_a^b f(x,y)\,dx$$

Both volumes represent $\iint_R f(x,y)\,dA$.

<u>Fubini's Thm</u>: If f is cont. on the rect. $R = [a,b] \times [c,d]$, then

$$\underbrace{\iint_R f(x,y)\,dA}_{\text{double integral}} = \underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy}_{\text{iterated integrals.}}$$

more general, this holds if f is bounded on R, f is discont. only on a finite number of smooth curves, and the iterated integrals exist.

ex2: calculate $I = \iint_R x\cos(x*y)\,dA$ over

$R = [0, \pi/6] \times [0, \pi/3]$

a) WRT y, then x

$$I = \int_0^{\pi/6} \underbrace{\int_0^{\pi/3} x\cos(x*y)\,dy}_{\left[\frac{x\sin(x*y)}{\cancel{x}}\right]_0^{\pi/3}}\,dx$$

$$= \int_0^{\pi/6} \sin\left(\frac{\pi}{3}x\right)\,dx$$

$$= \left[-\frac{3}{\pi}\cos\left(\frac{\pi}{3}x\right)\right]_0^{\pi/6}$$

$$= -\frac{3}{\pi}\left(\cos\left(\frac{\pi^2}{18}\right) - 1\right)$$

b) WRT x, then y.

$$I = \int_0^{\pi/3} \underbrace{\int_0^{\pi/6} x\cos(xy)\,dx}\,dy \qquad \begin{array}{ll} u = x & dv = \cos xy\,dx \\ du = 1 & v = \frac{1}{y}\sin(xy) \end{array}$$

$$\left[\frac{x\sin xy}{y} - \int \frac{1}{y}\sin xy\,dx\right]_0^{\pi/6}$$

Horrid!

$$\left[\frac{x\sin xy}{y} + \frac{1}{y^2}\cos(xy)\right]_0^{\pi/6}$$

$$= \int_0^{\pi/3}\left(\frac{\pi}{6y}\sin\left(\frac{\pi}{6}y\right) + \frac{36}{y^2}\cos\left(\frac{\pi}{6}\cdot y\right) - \frac{1}{y^2}\right)dy$$

Start w/ (b.) & use to motivate (a.)

$$= \frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\left(\frac{\pi}{6}y\right)}{y}\,dy + 36\int_0^{\pi/3} \frac{\cos\left(\frac{\pi}{6}y\right)}{y^2}\,dy$$

A handy trick

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

where $R = [a,b] \times [c,d]$

Q: why?

Look @ exercises in the text to show when it applies.

ex3: $\iint_R \frac{x}{1+xy}\,dA$ over $[0,1] \times [0,2]$

$$= \int_0^1 \underbrace{\int_0^2 \frac{x}{1+xy}\,dy}_{\left[\ln|1+xy|\right]_{y=0}^{y=2}}\,dx$$

$$= \int_0^1 \ln|1+2x|\,dx \qquad u = 1+2x, \quad \frac{du}{2} = dx$$

$$= \frac{1}{2}\int_1^3 \ln u\,du$$

$$= \frac{1}{2}(u\ln u - u)\Big|_1^3$$

$$= \frac{1}{2}(3\ln 3 - 3 + 1)$$