14.6: Directional Derive & the Gradient

Concept / picture of the directional derivative. (Mathematica).

$D_{\vec{u}}f \leftarrow$ a scalar

Relationship between contour plots & the gradient. (Mathematica).

$\nabla f \leftarrow$ a vector.

path of steepest ascent

$\boxed{\text{Def: The Gradient} \quad \nabla f = \langle f_x, f_y \rangle}$

Geometric derivation of the directional derivative (Mathematica & handout)

$$D_{\vec{u}}f = f_x \cos\theta + f_y \sin\theta$$

$$= \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle$$

Note that the direction $\vec{u}$ is given as a unit vector

$$= \nabla f \cdot \vec{u} \quad \text{(alg. form)}$$

$$= |\nabla f||\vec{u}| \cos\theta \leftarrow \text{angle between } \vec{u} \text{ \& } \nabla f.$$

$$= |\nabla f| \cos\theta \quad \text{(geo. formula)}.$$

The directional deriv. of $z = f(x,y)$ in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \quad \text{(alg. formula)}$$
$$= |\nabla f| \cos\theta \quad \text{(geo formula)}$$

the CCW angle made w/ the pos. x-axis.

While the derivation (handout) focused on the directional deriv., the main point is the <u>gradient ∇f</u>.

∇ prop. 1: The directional deriv. $D_{\vec{u}} f$ in any direction is the scalar projection of ∇f in that direction.

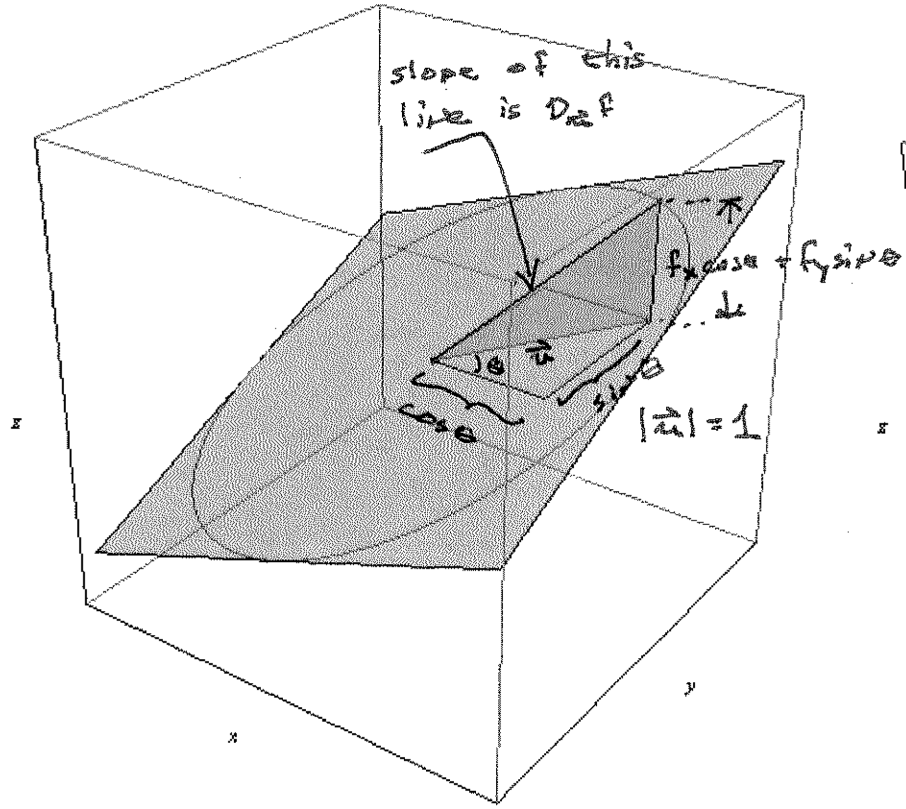
∇ prop 2: The vector ∇f points in the direction f increases in most rapidly.

∇ prop 3: The magnitude of ∇f is the max rate of increase of f

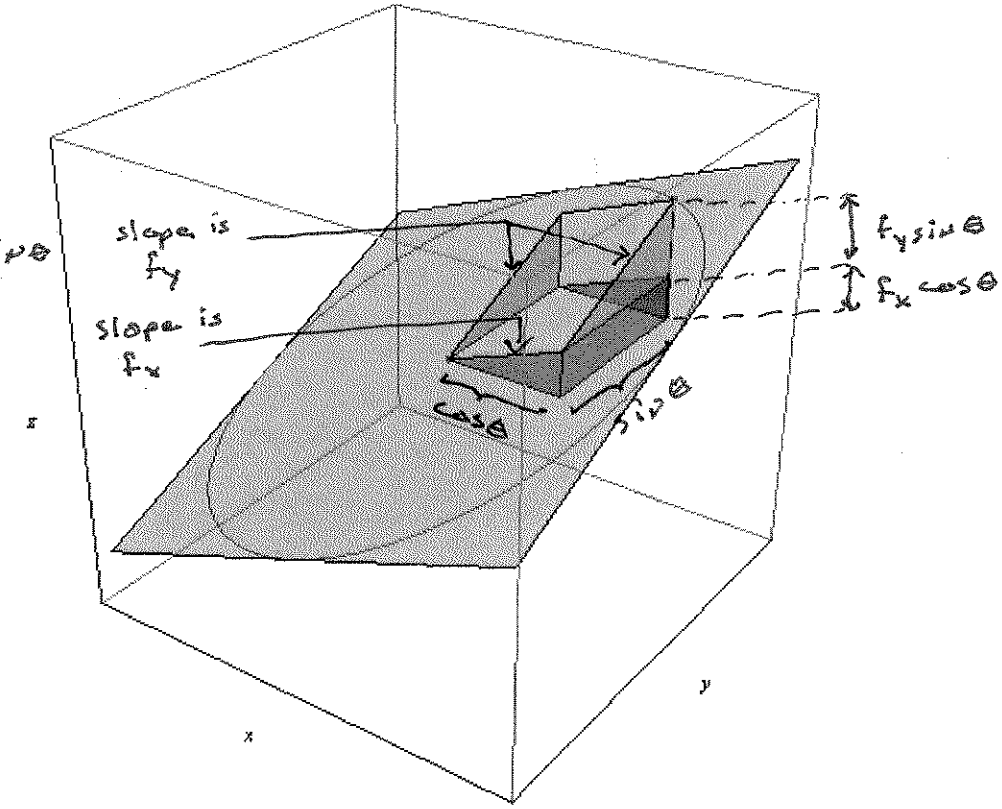
(explain 2 & 3 geometrically).

The connection between the directional and partial derivatives

Find the slope of the given triangle



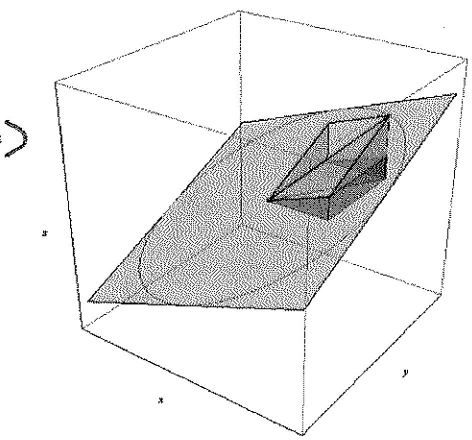
Intermediate step



$$D_u f = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f_x\cos\theta + f_y\sin\theta}{1}$$

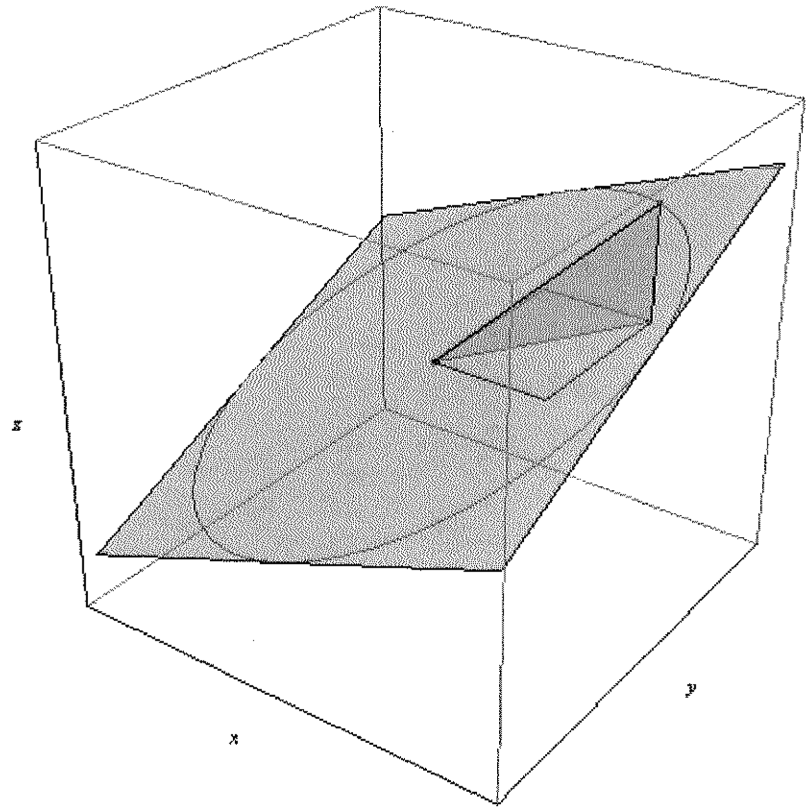
$$= \langle f_x, f_y\rangle \cdot (\cos\theta, \sin\theta)$$

The connection between the two pictures can be seen in the picture (to the right) where the diagrams are merged.

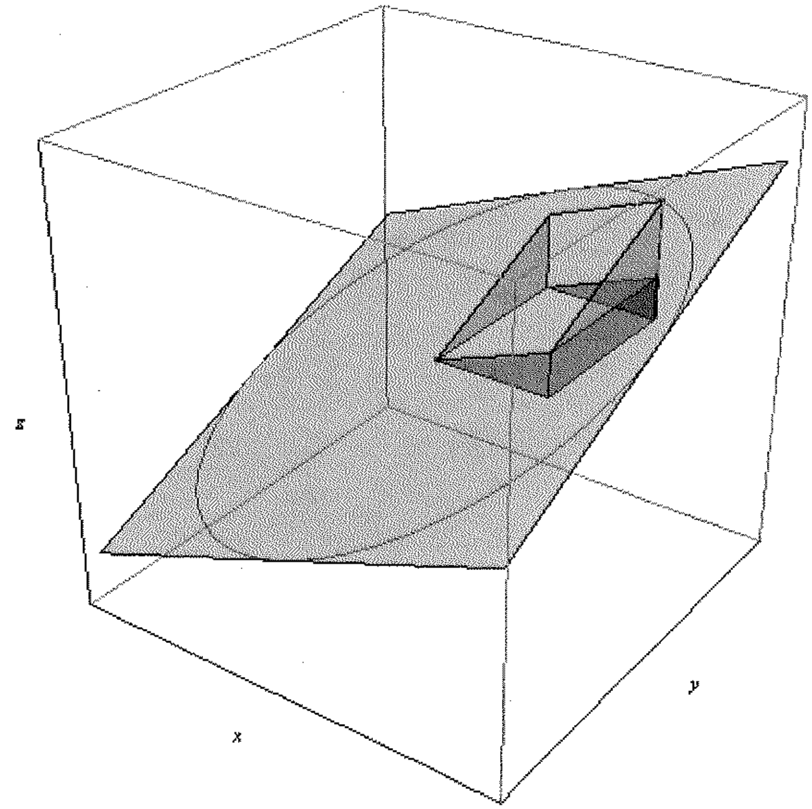


The connection between the directional and partial derivatives

Find the slope of the given triangle



Intermediate step



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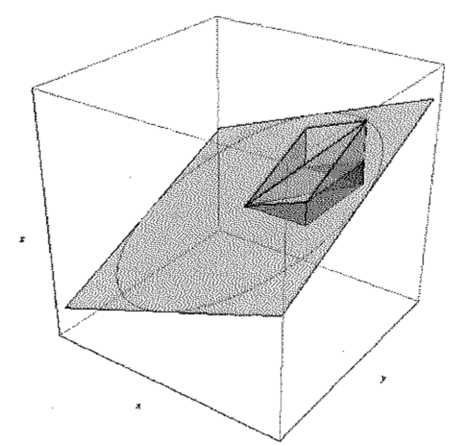

14.06
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Ex1: Find the directional deriv of the fct $f(x,y) = x^4 - x^2y^3$ @ $(2,1)$ in the direction of $\vec{u} = \langle 1, 3 \rangle$

In what direction(s) would the directional deriv be zero? maximized? (Mathematica).

How do we visualize $f(x,y,z) = w$ in $\mathbb{R}^4$?

If $k = w$ is constant, we have a level surface $f(x,y,z) = k$.

What does the gradient represent by analogy w/ the ∇f in $\mathbb{R}^3$.

Ex 2: Find the eqns of
(a) the tangent plane
(b) the normal line to...

$X - z = 4\arctan(yz)$ @ $P(1+\pi, 1, 1)$

$\Rightarrow \quad 0 = z - X + 4\arctan(yz)$
this corresponds to the level surface of $w = z - X + 4\arctan(yz)$ when $w = 0$.

(a) to find the plane, we need a normal vector $\Rightarrow \nabla w$.

$w_X = -1$

$w_y = \dfrac{4}{1+(yz)^2} \cdot z \Big|_P \quad 2$

$w_z = 1 + \dfrac{4y}{1+(yz)^2} \Big|_P \quad 3$

$\nabla w(1+p, 1, 1) = \langle -1, 2, 3 \rangle$ normal to the surface @ p

$\Rightarrow$ plane: $-1(X - (1+\pi)) + 2(y-1) + 3(z-1) = 0$

(b) the normal line

$\dfrac{x - (1+\pi)}{-1} = \dfrac{y-1}{2} = \dfrac{z-1}{3}$ symmetric eqts

$\vec{r}(t) = \langle 1+\pi, 1, 1 \rangle + t\langle -1, 2, 3 \rangle$ parametric eqt. for EWA