14.4: Tangent Planes & Linear Approx.

R.G

mathematica

Ex 1: Find the tangent plane to $f(x,y) = 2xy^3 - 5x^2$ @ $(3, 2, 3)$. Use the result to approx $z\big|_{(x,y)=(3.1,\,1.95)}$.

$$z_x = 2y^3 - 10x \Big|_{(3,2)} \quad -14$$

$$z_y = 6xy^2 \Big|_{(3,2)} \quad 72$$

$\Rightarrow$ The tangent plane is $T(x,y) - 3 = -14(x-3) + 72(y-2)$.

$$\Rightarrow f(3.1, 1.95) \approx T(3.1, 1.95)$$
$$\approx 3 - 14(0.1) + 72(-.05)$$
$$\approx -2$$

This is close to $f(3.1, 1.95)$ which is -2.077775.

It doesn't always work this well. (Mathematica)

Ex2: If $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{, else} \\ 0 & \text{, origin} \end{cases}$, we will have a hard time finding the tangent plane @ the origin.

Q: How do we know what we are in for?

We say that <u>differentiable</u> fcts can be approx by their tangent plane. So what does <u>differentiable</u> mean in a multivariable context.

<u>Dfn</u>: If $z=f(x,y)$, then f is differentiable @ (a,b) if Δz can be written as

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 & $\varepsilon_2 \mapsto 0$ as Δx & $\Delta y \mapsto 0$.

Horrible!

<u>Thm</u>: If the partials f_x & f_y exist near (a,b) & are cont. @ (a,b), then f is diff. @ (a,b).

(prf. in App. F).

Ex1 rev: (a) Show f is diff. @ $(3,2)$.

Duh – f_x, f_y exist & are cont. @ $(3,2)$

(b) Find the linear approx. & code for tangent plane.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$L(x,y) = 3 - 14(x-3) + 72(y-2)$$

Differentials

$y = f(x)$: $\quad dy = f'(x)\,dx \qquad$ (mathematical)

$z = f(x,y)$: $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

this is also called the <u>total differential</u>.

Q: How does this relate to the tangent plane?

Around (a,b), $f(x,y) \approx L(x,y)$

$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= f(a,b) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

or if $\Delta x = dx$ & $\Delta y = dy$:

$$= f(a,b) + dz$$

Hence $f(x,y) \approx f(a,b) + dz$.

Ex 3: The length & width of a rect. are measured @ 18 in. & 27 in. respectively w/ an error in measurement of @ most 0.1 in in length & 0.05 in in width. Use differentials to est. the max error in the calculated area.

$$A(L, W) = LW$$

$$\Rightarrow A_L = W \;;\; A_W = L$$

$$\Rightarrow dA = W \cdot dL + L \cdot dW$$

and the max is $dA = 27(.1) + 18(.05)$

$= 3.6 \text{ in}^2.$

Note! there is error in the text on p 898 where it misses the pt that dz is an estimate. (this note applied to the 6th ed).