8.2 : Homogeneous Systems.

A dynamical system

A stretch of desert in northwestern Mexico is populated mainly by two species of animals: coyotes and roadrunners. We wish to model the populations $c(t)$ and $r(t)$ of coyotes and roadrunners t years from now if the current populations c_0 and r_0 are known.[1,2]

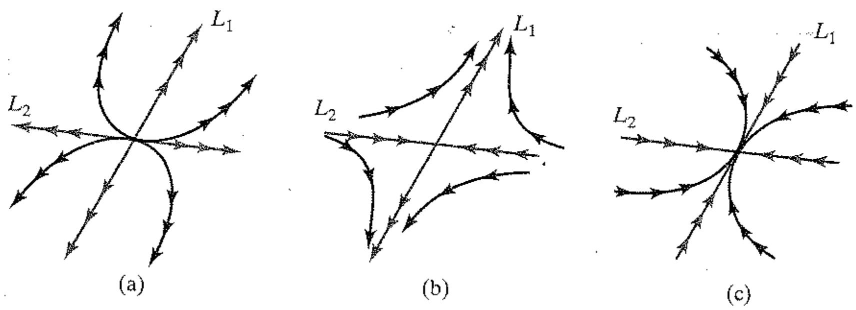

Figure 11 (a) $\lambda_1 > \lambda_2 > 1$. (b) $\lambda_1 > 1 > \lambda_2 > 0$. (c) $1 > \lambda_1 > \lambda_2 > 0$.

The general homogeneous linear 1st. order system is $X' = AX$. where $A_{n \times n}$ is a matrix of constants.

We want to find solns. of the form:

$$X = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t} = K e^{\lambda t}$$

If such soln. exist, we call λ an eigenvalue and X an eigenvalue.

Derivation.

If $\underline{X} = \underline{k}e^{\lambda t}$ is a soln. to $\underline{X}' = A\underline{X}$,

then $\underline{k}\lambda e^{\lambda t} = A\underline{k}e^{\lambda t}$

$\Rightarrow \underline{k}\lambda = A\underline{k}$

$\Rightarrow A\underline{k} - \lambda\underline{k} = \vec{0}$

$\Rightarrow A\underline{k} - \lambda I\underline{k} = \vec{0}$

$\Rightarrow (A - \lambda I)\underline{k} = \vec{0}$

obviously $\underline{k} = \vec{0}$ is a soln., but we want <u>trivial solns.</u> to the homogeneous eqn.

To find trivial solns, we need the cols of $(A - \lambda I)$ to be L.D. ... that is $\det(A - \lambda I) = 0$

once you find the eigenvalues, RREF $[A - \lambda I \,\vdots\, \vec{0}]$ to find the corresponding eigenvector(s).

<u>CASE 1:</u> UNIQUE REAL λ.

<u>ex 1:</u>
$$\begin{aligned} x' &= 2x + y \\ y' &= y \end{aligned} \Rightarrow \underline{X}' = \overbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}^{A}\underline{X}$$

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 1, \lambda = 2$

$\lambda = 1$: $A - I = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 2$: $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \longrightarrow k_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So $\underline{X} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$

ex2: $X' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} X$

solve $\det(A - \lambda I) = 0$

$$\Rightarrow \quad 0 = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\Big[\underbrace{(2-\lambda)(-1-\lambda)-3}_{\lambda^2-\lambda-5}\Big] + (1+\lambda)$$

$$= (-1-\lambda)\left[\lambda^2 - \lambda - 6\right]$$

$$= (-1-\lambda)(\lambda-3)(\lambda+2)$$

so the eigenvalues are $-1, -2, 3$

$\lambda = -1$: $\begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow K_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -2$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow K_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow K_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

so $X = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t}$

CASE 2: λ w/ multiplicity.

Eigenvalues are not necessarily unique. λ w/ multiplicity > 1 fall into 2 categories.

(A) $A_{n \times n}$ has n eigenvectors.

ex3: $\mathbf{X}' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \mathbf{X}$ (symmetric)

w/ eigenvalues $\lambda = 3$ (mult 2) and -3

$$\lambda = 3: \quad \begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$\lambda = -3: \quad \begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvector $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

These are eigenvectors of A and so the fundamental set is:

$$\mathbf{X}_1 = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{X}_2 = c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{X}_3 = c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{X} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(B) $A_{n\times n}$ has less than n eigenvectors

ex4: $\mathbf{X}' = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \mathbf{X}$

$\lambda = 2$ (mult 2)

$\lambda = 2$: $\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ eigenvector $\mathbf{K} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since there is only 1 eigenvector, we need to find $\mathbf{P}$ s.t. $(A - \lambda I)^2 \mathbf{P} = \vec{0}$.

That is $(A - \lambda I)\mathbf{K} = \vec{0}$ $\boxed{\text{Derivation on p 339-340}}$

and $(A - \lambda I)^2 \mathbf{P} = \vec{0}$ (requires two steps to get to $\vec{0}$)

so solve $[A - \lambda I \mid \mathbf{K}]$

$$\operatorname{rref}\left(\begin{bmatrix} -3 & 3 & \vdots & 1 \\ -3 & 3 & \vdots & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 & \vdots & -1/3 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

so $\mathbf{P} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$

$Y_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Y_2 = t e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$

$$\mathbf{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} e^{2t} \right]$$

CASE 3: Complex λ

I'm gonna take a slightly different approach from the text.

(0) Start: $\underline{X}' = A\underline{X}$, $A_{n\times n}$

(1) $\lambda = \alpha \pm \beta i$ are eigenvalues w/ corresponding eigenvectors $\underline{K}$ & $\overline{\underline{K}}$

$\underline{X}_1 = e^{\lambda t}\underline{K} = e^{(\alpha+i\beta)t}(\vec{a}+i\vec{b})$

$\underline{X}_2 = e^{\bar{\lambda}t}\overline{\underline{K}} = e^{(\alpha-i\beta)t}(\vec{a}-i\vec{b})$ where $\underline{K} = \vec{a}+i\vec{b}$

(2) Focus on $\underline{X}_1$

$$\underline{X}_1 = e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)(\vec{a}+i\vec{b})$$

$$= e^{\alpha t}\left((\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b}) + i(\sin\beta t\,\vec{a} + \cos\beta t\,\vec{b})\right)$$

$$= \vec{x}_1 + i\vec{x}_2$$

(3) recall $\underline{X}_1$ is a soln:

$$\underline{X}_1' = A\underline{X}_1$$

$$\Rightarrow \vec{x}_1' + i\vec{x}_2' = A\vec{x}_1 + iA\vec{x}_2$$

$$\Rightarrow \vec{x}_1' = A\vec{x}_1 \quad \text{and} \quad \vec{x}_2' = A\vec{x}_2$$

(4) The soln:

$$\vec{x}_1 = e^{\alpha t}(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b})$$

$$\vec{x}_2 = e^{\alpha t}(\sin\beta t\,\vec{a} + \cos\beta t\,\vec{b})$$

exs: Find a general soln. of $\underline{X}' = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} \underline{X}$

Find the eigenvalues.

$$0 = \begin{vmatrix} -2-\lambda & -5 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= -(2+\lambda)(2-\lambda) + 5$$

$$= \lambda^2 + 1$$

$$\Rightarrow \lambda = \pm i$$

Find one complex eigenvector (for $\lambda = -i$)

$= 0 \pm i(-1)$

$$A + i I = \begin{bmatrix} -2+i & -5 \\ 1 & 2+i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+i \\ 1 & 2+i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+i \\ 0 & 0 \end{bmatrix}$$

eigenvector is $\begin{bmatrix} -2-i \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Fundamental set.

$$\underline{X}_1 = e^{0t} \cos(-t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} - e^{0t} \sin(-t) \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\underline{X}_2 = \sin(-t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \cos(-t) \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and $\underline{X} = c_1 \left(\sin(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \cos(t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$

$$+ c_2 \left(\cos(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \sin(t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$$

$\boxed{\begin{matrix} 8.2 \\ 8/8 \end{matrix}}$

ex 6: Find the general soln. of $\underline{X}' = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \underline{X}$

Find the eigenvalues

$$0 = \begin{vmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 4\lambda + 5$$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$= -2 \pm i$$

Find the eigenvector corresponding to $\lambda = -2+i$

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1+i \\ -1 & -1-i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & + & 1+i \\ 0 & & 0 \end{bmatrix}$$

w/ eigenvector: $\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The fundamental set:

$$\underline{X}_1 = e^{-2t}\left(\cos t \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$\underline{X}_2 = e^{-2t}\left(\sin t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$