7.3.1: Translation on the s-Axis

recall: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$\Rightarrow \mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

In other words, multiplying $f(t)$ by e^{at} shifts F a units right.

ex1: (a) $\mathcal{L}\{t^6\}$

(b) $\mathcal{L}\{e^{-2t} t^6\} = \mathcal{L}\{t^6\}\Big|_{s \to s+2}$

(c) $\mathcal{L}\{\sin 3t\}$

(d) $\mathcal{L}\{e^{4t} \sin 3t\} = \mathcal{L}\{\sin 3t\}\Big|_{s \to s-4}$

ex2: (a) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

(b) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s \to s+2}\right\}$$

$$= e^{-2t} \cos t.$$

ex3: $y'' + 2y' + y = 0 \quad , \quad y(0) = 1, \; y'(0) = 1$

$$\Rightarrow \quad s^2 Y(s) - s\,y(0) - y'(0) + 2(s\,Y(s) - y(0)) + Y(s) = 0$$

$$\Rightarrow \quad Y(s)(s^2 + 2s + 1) = s + 3$$

$$\Rightarrow \quad Y(s) = \frac{s+1}{s^2+2s+1}$$

$$= \frac{s+1}{(s+1)^2}$$

$$= \frac{1}{s+1}$$

$$\Rightarrow \quad y(t) = e^{-t} \cdot 1$$

ex4: $y'' - 6y' + 9y = t \quad , \quad y(0) = 0 \; , \; y'(0) = 1$

$$\Rightarrow \quad s^2 Y(s) - s\,y(0) - y'(0) - 6(s\,Y(s) - y(0)) + 9\,Y(s) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s)(s^2 - 6s + 9) = \frac{1}{s^2} + 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2(s-3)^2} + \frac{1}{(s-3)^2}$$

$$= \frac{2/27}{s} + \frac{1/9}{s^2} - \frac{2/27}{(s-3)} + \frac{1/9}{(s-3)^2} + \frac{1}{(s-3)^2}$$

$$\Rightarrow \quad y(t) = \frac{2}{27} + \frac{1}{9}t - e^{3t} \cdot \frac{2}{27} + \frac{10}{9} e^{3t}\, t$$

ex5: $y'' - 2y' - y = e^{2t} - e^{t}$, $y(0) = 1$, $y'(0) = 3$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) - 2\big(s\,Y(s) - y(0)\big) - Y(s) = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\Rightarrow Y(s)\left(s^2 - 2s - 1\right) = \frac{1}{s-2} - \frac{1}{s-1} + s + 1$$

$$\Rightarrow Y(s) = \frac{1}{(s-2)(s^2-2s-1)} - \frac{1}{(s-1)(s^2-2s-1)} + \frac{s+1}{s^2-2s-1}$$

$$= \frac{(s-1) - (s-2) + (s+1)(s-1)(s-2)}{(s-1)(s-2)(s^2-2s-1)}$$

$$= \frac{s^3 - 2s^2 - s + 3}{(s-1)(s-2)(s^2-2s-1)}$$

$$= \frac{-1/4}{s-1} + \frac{-1/4}{s+1} + \frac{3}{2} \cdot \underbrace{\frac{s-2}{s^2-2s-1}}_{\frac{\sqrt{2}}{\sqrt{2}}\,\frac{(s-1)-1}{(s-1)^2-2}}$$

$\leftarrow$ use cosh or sinh

$$\Rightarrow y(t) = -\frac{1}{4}e^{t} - \frac{1}{4}e^{-t} + \frac{3}{2}\,e^{t}\cosh\sqrt{2}t - \frac{3}{2\sqrt{2}}\,e^{t}\sinh\sqrt{2}t$$