7.2: Inv. Transforms and Transforms of Derivatives.

so how does the Laplace transform help us solve D.E. ?

$$
\begin{tikzcd}
\text{DE in } t \arrow[r] \arrow[d, "\mathcal{L}"'] & \text{soln in } t \\
\text{DE in } s \arrow[r, "\text{solve w/s}"'] & \text{soln in } s \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}
$$

__ex1:__ DE in t $\Leftrightarrow$ a DE in s.

$$y' - y = 1, \quad y(0) = 0$$

$$\Rightarrow \mathcal{L}\{y' - y\} = \mathcal{L}\{1\}$$

$$\Rightarrow \mathcal{L}\{y'\} - \mathcal{L}\{y\} = \frac{1}{s}$$ key: our soln is $y(t)$

$$\Rightarrow \mathcal{L}\{y'\} - Y(s) = \frac{1}{s}$$

we need $\mathcal{L}\{y'\} = \int_0^\infty e^{-st} y'(t)\, dt$

$u = e^{-st}$

$du = -s e^{-st} dt$

$dv = y'(t)\, dt$

$v = y(t)$

$$= \left[e^{-st} y(t) \quad s \int e^{-st} y \, dt\right]_{t=0}^{t=\infty}$$

$$= s\,Y(s) - y(0)$$

$$\Rightarrow s\,Y(s) - \underbrace{y(0)}_{0} - Y(s) = \frac{1}{s}$$

$$\Rightarrow \quad Y(s)(s-1) = \frac{1}{s}$$

$$\Rightarrow \quad Y(s) = \frac{1}{s(s-1)}$$

Notice : Our goal is to find $y(t)$.

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\Rightarrow \quad \mathcal{L}\{y(t)\} = \frac{1}{s(s-1)}$$

So to solve the DE, we just need to find the $y(t)$ whose Laplace transform is $\frac{1}{s(s-1)}$.

That is, we've solved the DE in s!

The last step: $\frac{1}{s(s-1)} = \frac{-1}{s} + \frac{1}{s-1}$

$$\mathcal{L}\{y(t)\} = \frac{1}{s-1} - \frac{1}{s}$$

$$\Rightarrow \quad y(t) = e^{t} - 1$$

There are two hard parts of this:

(1) $\mathcal{L}\{y'\}$

(2) finding $y(t)$ given $Y(s)$

Let's tackle these one @ a time.

(1) $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

$$\vdots$$

$$\mathcal{L}\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y(0) - \dots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

Let's derive the 2^{nd} deriv. formula.

$$\mathcal{L}\{y''(t)\} = \int_0^\infty e^{-st}\, y''(t)\,dt$$

$u = e^{-st}$

$du = -se^{-st}dt$

$dv = y''(t)dt$

$v = y'(t)$

$$= \left[e^{-st}y'(t) + s\int e^{-st}y'(t)\,dt\right]_0^\infty$$

$$= s\underbrace{\int_0^\infty e^{-st}y'(t)\,dt}_{\mathcal{L}\{y'(t)\}} - y'(0)$$

$$= s\{sY(s) - y(0)\} - y'(0)$$

$$= s^2Y(s) - sy(0) - y'(0)$$

<u>ex 2</u>: Find $Y(s)$ if $y'' + 5y' + 4y = 0$, $y(0) = 1$, $y'(0) = -1$

<u>ex3</u>: Find $Y(s)$ if $2y''' + 3y'' - 3y' - 2y = e^{-t}$, $y(0) = y'(0) = 0$ and $y''(0) = 1$.

ex4: $y'' - y' - 6y = 0$, $y(0) = 2$, $y'(0) = -1$

$$\Rightarrow Y(s) = \frac{2s-3}{s^2-s-6} = \frac{3/5}{s-3} + \frac{7/5}{s+2}$$

$$\Rightarrow y = \frac{3}{5}e^{-3t} + \frac{7}{5}e^{-2t}$$

ex5: $y'' - 2y' + 5y = -8e^{-t}$, $y(0) = 2$, $y'(0) = 12$

$$\Rightarrow Y(s) = \frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{3(s-1) + 2(4)}{(s-1)^2+2^2} - \frac{1}{s+1}$$

(requires a shift)

$$\Rightarrow y = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}$$

ex6: $y'' + 4y' - 5y = te^{t}$, $y(0) = 1$, $y'(0) = 0$

$$\Rightarrow Y(s) = \frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3} = \frac{35/216}{s+5} + \frac{181/216}{s-1} + \frac{-1/36}{(s-1)^2} + \frac{1/6}{(s-1)^3}$$

(requires a shift).

$$\Rightarrow y = \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}te^{t} + \frac{1}{12}t^2e^{t}$$